IMPINGING OF SLOW STAGNATION FLOWS NEAR A SLIT

[1]HYUNG JONG KO, [2]KYOUNG HOON KIM

Department of Mechanical Engineering, Kumoh National Institute of Technology, Korea
Email: kohj@kumoh.ac.kr, khkim@kumoh.ac.kr

Abstract—Two-dimensional impinging of slow stagnation flows near a slit is investigated using the Stokes approximation. For the arbitrary combination of their strengths and centerline shifts, the flow is described using two complex analytic functions which are analytically determined so as to satisfy the boundary conditions. The streamline patterns for the representative cases, the flow behavior near the sharp edges, and the stress distributions on the plates are calculated. The results show that the flow separates horizontally from the edges and the saddle stagnation point lies on the slit or on the side of the plates.

Index Terms—Complex Velocity, Impinging, Stagnation Flow, Stokes Approximation.

I. INTRODUCTION

Because of the nonlinearity, exact solutions of the Navier-Stokes equation are rare and classified into two groups: unidirectional flow solutions and similarity solutions. One of the latter group which has attracted much attention is the two-dimensional stagnation point flow frequently referred to as Hiemenz flow [1]. This flow is realized when a fluid stream impinges a plane boundary. The boundary may be curved like the surface of a circular cylinder if the radius of curvature is large enough [2].

This implies that many flows near a front side of a blunt body placed in a moving fluid can be approximated by this flow. A review and extension of the similarity stagnation point solutions of the Navier–Stokes equations is given by Wang [3]. There have been studies on the two-dimensional stagnation flows toward a plane wall having a modified shape. Suh and Liu [4] numerically considered a stagnation flow around a vertical plate attached on a plane wall.

The same flow was analytically considered by Jeong and Ko [5] using the Stokes approximation. Later Jeong [6] extended the analysis such that the vertical plate was allowed to be detached from the plane wall. The case in which a plane wall is replaced by a flat plate with a slit was investigated by Ko and Jeong [7] using the Stokes approximation. Recently Ko and Han [8] investigated the effect of centerline shift on this flow. There have been several studies on the two-dimensional slow viscous flow around a slit. Hasimoto [9] considered the flow through a slit due to a pressure difference using an associated potential flow solution. Jeong and Kim [10] generalized this flow and investigated the flow in a converging nozzle. Ko and Kim [11] studied a two-dimensional slow viscous flow due to a Stokeslet near a slit. This flow is regarded as a first approximation to the viscous flow due to the translation of a small circular cylinder near a crack of flat plate.

In this study, two-dimensional slow viscous flow due to the impinging stagnation flows near a slit is investigated using the Stokes approximation. This flow is different from the flow studied by Wang [12] in that the same fluid impinges in the presence of sold boundary. It is realized in the vicinity of a flat plate with a small crack when two fluid streams coming from each side of the plate stagnate near the crack. Both stagnation flows are allowed to have different flow strength and centerline shift. The flow is first described by two complex functions and then analytically obtained by solving a pair of Riemann-Hilbert problems so as to satisfy the boundary conditions.

For the varying values of the strengths and centerline shifts, the streamline patterns are numerically determined using a closed form solution. The local flow behavior near the sharp edges, the pressure and the shear stress distributions on the plate, and the location of the saddle stagnation point are also calculated.

II. FORMULATION AND ANALYSIS

Two-dimensional slow viscous impinging flow sketched in Fig. 1 is investigated, in which two stagnation flows from infinities above and below the horizontal plates approach toward a slit with a length of 2. The upper and the lower stagnation flows have a strength of K_1 and K_2, and a centerline shift of s_1 and s_2, respectively. The origin of the Cartesian coordinates (x, y) is at the center of the slit and the plates are on the x axis.

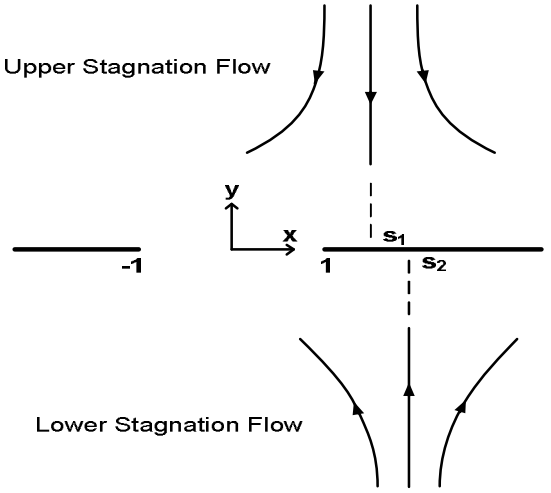


Fig. 1. Sketch of the flow field.

The governing equations for the velocity (*u*, *v*) and the pressure *p* under the Stokes approximation are given as

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial p}{\partial x}, \quad (1)$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial p}{\partial y}, \quad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\ . \quad (3)$$

Here μ is the coefficient of viscosity of the fluid. The complex velocity *W* and *p* satisfying the equations (1)-(3) can be written using two complex functions of a complex variable $z=x+iy$ as follows [13].

$$W=u-iv=F(z)+(\bar{z}-z)\frac{dG}{dz}-\overline{G(z)}, \quad (4)$$

$$p-i\mu\Omega=4\mu\frac{dG}{dz}. \quad (5)$$

Here overbar denotes the complex conjugate and Ω is the vorticity component of the fluid normal to the *x*-*y* plane and the complex functions *F*(*z*) and *G*(*z*) must be analytic in the whole *z*-plane except the flat plates. In this case, the Schwarz reflection principle and the symmetry of the flow region with respect to the x axis ensure that the conjugate function $\overline{G}(z)=\overline{G(\bar{z})}$ is also analytic in the whole flow region [14].

With reference to Fig. 1, the velocity must satisfy the following no-slip and far-field boundary conditions:

$$W=0 \ \ at \ \ y=0, \ |x|\geq 1\ , \quad (6)$$

$$W\rightarrow\begin{cases}K_1 y(2x-2s_1+iy) & as \ \ y\rightarrow\infty\\ -K_2 y(2x-2s_2+iy) & as \ \ y\rightarrow-\infty\end{cases}. \quad (7)$$

As a first step to find *F* and *G* satisfying (7) and (8), the limiting behaviors of these functions on the upper and lower sides of the plates are to be examined. Denoting the limiting value of a function on the upper/lower side by the subscript +/- and using (4), the following relations between the limiting values result from (6).

$$[F]_+-[\overline{G}]_-=0\ , \quad (8)$$

$$[F]_--[\overline{G}]_+=0\ . \quad (9)$$

Subtracting (9) from (8) and adding (9) to (8), the following relations between the boundary values are obtained.

$$[F+\overline{G}]_+-[F+\overline{G}]_-=0\ , \quad (10)$$

$$[F-\overline{G}]_++[F-\overline{G}]_-=0\ . \quad (11)$$

Since a linear combination of analytic functions is also analytic, (10) and (11) are the Riemann-Hilbert problems for the analytic functions $F(z)+\overline{G}(z)$ and $F(z)-\overline{G}(z)$, respectively. To satisfy the condition (7), these functions should not grow faster than a quadratic polynomial of z at infinity. In addition, $F-\overline{G}$ must ensure such that the velocity is finite at the sharp edges ($z=\pm1$). Because of this requirement, the solution showing the lowest singularity at the sharp edges should be taken from all possible solutions. A set of general solutions of (10) and (11) satisfying these conditions are written as follows [14].

$$F(z)+\overline{G}(z)=A_0+A_1 z+A_2 z^2\ , \quad (12)$$

$$F(z)-\overline{G}(z)=\sqrt{z^2-1}\left(B_0+B_1 z\right). \quad (13)$$

The unknown complex constants A_0, B_0, etc. contained here are determined so as to exactly satisfy the condition (7). The function $k(z)=\sqrt{z^2-1}$ is made single valued by restricting the arguments as

$$-\pi<\arg(z+1)<\pi,\ \ 0<\arg(z-1)<2\pi\ . \quad (14)$$

This choice is physically reasonable because it is identical to that $k(z)$ takes the two flat plates as branch cuts. It is worthy to note that $\overline{k}(z)=-k(z)$. It is not difficult to solve for the undetermined constants in (12) and (13) so as to satisfy the condition (7) by way of (4). The final solutions are as follows:

$$F(z)=-(iK_1/8)(z-2s_1)[z+k(z)]+(iK_2/8)(z-2s_2)[z-k(z)], \quad (15)$$

$$G(z)=(iK_1/8)(z-2s_1)[z+k(z)]-(iK_2/8)(z-2s_2)[z-k(z)]. \quad (16)$$

As the two functions describing the flow have been obtained, the flow variables such as velocity, pressure, and vorticity can be readily calculated.

RESULTS AND DISCUSSIONS

The functions *F* and *G* obtained in a closed form are to be used to describe the two-dimensional viscous flow under consideration. The streamline patterns are easily explored by using a stream function which automatically satisfies the continuity equation (3). The expression for the conventional stream function Ψ which is compatible with *W* in (4) is as follows [13]:

$$\Psi = \mathrm{Im}\left[\int F(z) + G(z)dz + (\bar{z} - z)G(z)\right] \qquad (17)$$

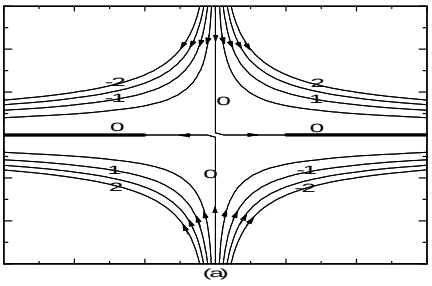
Here "Im" means the "imaginary part." Substituting (15) and (16) into (4) and (17), the following expressions for the complex velocity and the stream function are obtained:

$$\begin{aligned} W = {} & (K_1 y/4)[z/k(z)+1]\cdot[z-2s_1+k(z)] \\ & + (K_1/4)\,\mathrm{Im}\{(z-2s_1)[z+k(z)]\} \\ & + (K_2 y/4)[z/k(z)-1]\cdot[z-2s_2-k(z)] \\ & - (K_2/4)\,\mathrm{Im}\{(z-2s_2)[z-k(z)]\}, \end{aligned} \qquad (18)$$

$$\begin{aligned} \Psi = {} & (K_1 y/4)\,\mathrm{Im}\{(z-2s_1)[z+k(z)]\} \\ & - (K_2 y/4)\,\mathrm{Im}\{(z-2s_2)[z-k(z)]\}. \end{aligned} \qquad (19)$$

For any values of K_1, K_2, s_1, and s_2, (19) shows that Ψ = 0 for y = 0, i.e. on the x-axis. This implies that the x-axis always becomes one of the streamlines and there is no flow across the slit. It follows that the volume flowrate through the slit is zero. It also follows that the flow separates horizontally from the edge of the flat plates.

The streamline patterns obtained by numerically calculating the stream function are shown in Fig. 2 to Fig. 4 for the representative set of values of the strength and centerline shift in a domain of [-3, 3]×[-3, 3]. The streamlines corresponding to the values of Ψ between -2 and 2 are plotted with an increment of 0.5 and those are shown for every second line. Because of the geometric symmetry with respect to the y- axis and the linearity of flow, only the cases where s_1 is nonnegative are presented. However s_2 is allowed to be negative. The flow patterns for negative s_1 can be acquired by speculating that of $-s_1$ with the same values of other parameters.



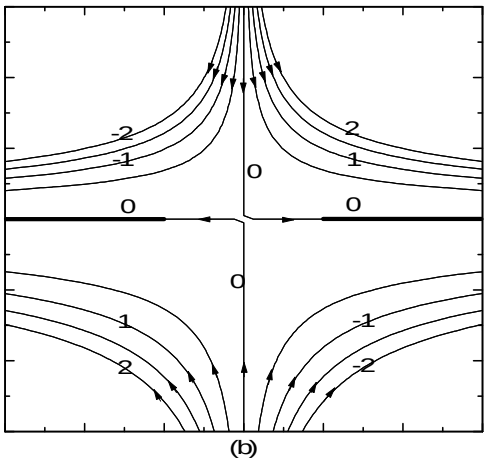
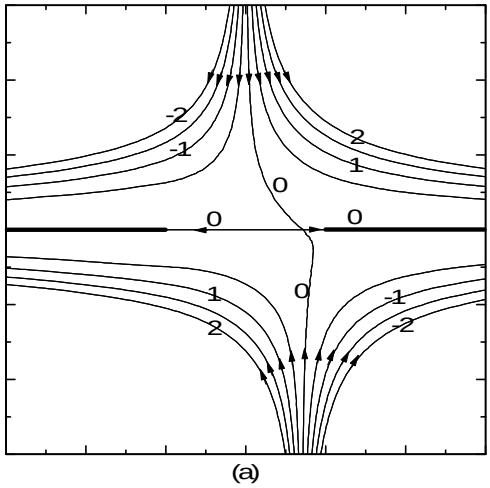


Fig. 2. Streamlines for the cases without centerline shift; (a) K_1 = 1, K_2 = 1, (b) K_1 = 1, K_2 = 0.3. The numbers denote the values of Ψ.

The flow patterns for the case where both stagnation flows have no centerline shift are shown in Fig. 2 (a) for the same strengths and in Fig. 2 (b) for the different strengths. In both cases, the saddle stagnation point is at the center of the slit and the flow is symmetric with respect to the y-axis. For the same strength in (a): K_1 = K_2 = 1, the flow is also symmetric with respect to the x-axis. For the different strength in (b): K_1 =1, K_2 = 0.3, gap between neighboring streamlines in the lower half plane is about three times wide than that of the upper half plane. This is because the flow in the lower half plane about 0.3 times slow than in the upper half plane. Note that velocity is inversely proportional to the gap distance between the adjacent streamlines for the same increment of the stream function. Since the flow region is symmetric with respect to the x-axis, the flow pattern of the opposite case of K_1<K_2 can be inferred from the flow obtained by exchanging K_1 and K_2 and appropriately rescaling the stream function.



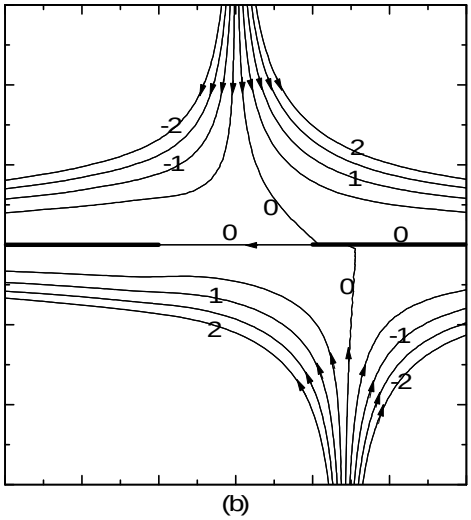


Fig. 3. Streamlines for the case where K_1 = 1, K_2 = 1, and the lower stagnation flow only has a centerline shift; (a) s_1 = 0, s_2 = 0.7, (b) s_1 = 0, s_2 = 1.4. The numbers denote the values of Ψ.

The flow patterns when one of the stagnation flow has a centerline shift are shown in Fig. 3. As the strengths K_1 and K_2 affect the flow only quantitatively, both of them are taken to be 1. Owing to the linearity of flow and the geometric symmetry, it is sufficient to investigate the case in which the centerline shift exists in the lower flow. With $s_1 = 0$, two representative cases are presented for $s_2 = 0.7$ and 1.4 in Fig. 3 (a) and Fig. 3 (b). The flows are no longer symmetric with respect to the x-axis. For a small value of s_2, saddle stagnation point still lies on the slit. If s_2 is large enough to sweep the flow to the right, then each flow stagnate onto the flat plate. The flow patterns corresponding to the negative values of s_2 can be conjectured by considering the flow for $-s_2$ and taking its mirror image about the y-axis. This inference is also applicable to the cases where K_1 and K_2 differ each other.

The flow patterns representing the cases where both upper and lower stagnation flows have a shifted centerline are shown in Fig. 4. The same values of the parameters with $K_1 = K_2 = 1$, and $s_1 = 0.7$ are used in (a), (b), and (c). While s_2 is varied from -1.7 in (a) to 1.4 in (c). Flow symmetry is possible if s_2 is properly selected. Specifically, it has a point symmetry about the origin if $s_2 = -0.7$ as seen in Fig. 4 (b). It will be symmetric with respect to the x-axis if $s_2 = 0.7$. In Fig. 4 (a) the upper flow stagnates at the left edge, while the lower one onto the lower side of the left plate. Adjusted case of Fig. 4 (b) has a saddle stagnation point at the center of the slit, i.e. at the origin. For sufficiently large value of s_2 in Fig. 4 (c), the upper and lower flows stagnate onto the upper and lower side of the right plate. The flow pattern for $s_2 = 0$ can be conjectured from the mirror image of that in Fig. 3 (a) with respect to the x-axis.

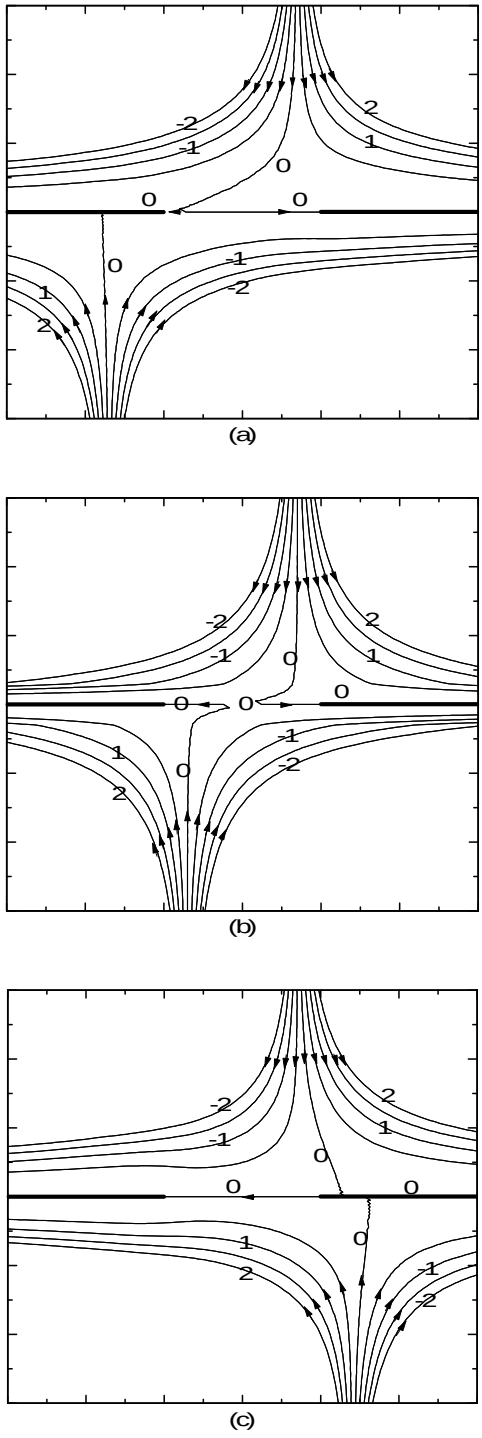


Fig. 4. Streamlines for the case where K_1 = 1, K_2 = 1, and both stagnation flows have a centerline shift; (a) s_1 = 0.7, s_2 = -1.7, (b) s_1 = 0.7, s_2 = -0.7, (c) s_1 = 0.7, s_2 = 1.4. The numbers denote the values of Ψ.

Up to here the representative cases of the flow were investigated using the numerically computed stream function. All of the flow patterns are topologically same. There exists no viscous eddy which has been found in a stagnation flow toward a slit over a quiescent flow [7, 8]. Depending on the parameter values, the impinging flows toward a slit either

stagnate at one saddle stagnation point located on the slit or on the sides of the flat plates.

In order to further investigate the separation of the flow from the flat plate, the flow near the edge is closely examined. From a geometric symmetry, it is sufficient to analyze for the right edge. After a little calculation, the following asymptotic expansion of the stream function in the neighborhood of $z = 1$ is obtained for a local polar coordinate (R, ϕ):

$$z = 1 + Re^{i\phi}, \quad (R \to 0, 0 < \phi < 2\pi) \quad . \tag{20}$$

$$\psi \approx \sqrt{\tfrac{1}{2}}\left[K_1(1-2s_1) + K_2(1-2s_2)\right] R^{3/2} \cos\left(\tfrac{\phi}{2}\right) \sin^2\left(\tfrac{\phi}{2}\right) \quad . \tag{21}$$

This is a form which agrees with that of the general behavior of the Stokes flow near a sharp edge [15]. From (21) it is evident that the flow separation angle from the right edge is $\phi = \pi$; that is, the flow separates to the left from (or attaches to the right to) the right edge.

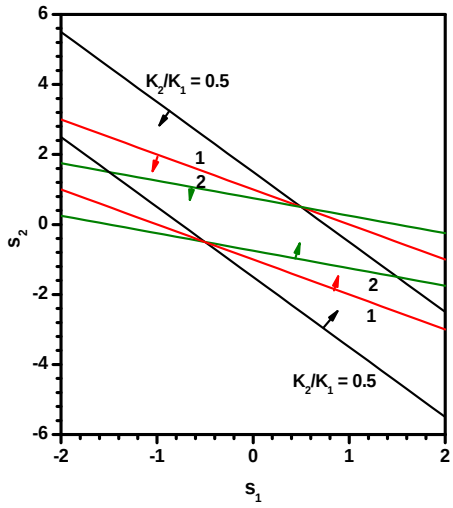


Fig. 5. The range of s_1 and s_2 for which a saddle stagnation point lies on the slit for some ratios of K_1 and K_2. The upward and downward arrows denote the lower and upper limits, respectively.

The condition for the location of saddle stagnation point is obtained by exploring the velocity distribution on the x-axis. Direct calculation of W of (18) for $y = 0$ leads to

$$W = \begin{cases} \frac{1}{4}\sqrt{1-x^2}\left[K_1(x-2s_1) + K_2(x-2s_2)\right] & (|x| < 1) \\ 0 \quad (|x| > 1) \end{cases} \quad . \tag{22}$$

This means that a saddle stagnation point locates on the slit at $x = 2\,(K_1 s_1 + K_2 s_2) / (K_1 + K_2)$. As this value must be between -1 and 1, there is a range of s_1 and s_2 where a stagnation point exists for given K_1 and K_2. It is dependent on the ratio of K_1 and K_2 and given pictorially in Fig. 5 for some representative values of this ratio. For example, for $s_1 = 0$ and $K_2/K_1 = 1$, a saddle stagnation point exists if $-1 < s_2 < 1$ (See Figs. 2 and 3).

Finally the pressure and the vorticity fields and the stress distributions on the plates are investigated. From (5) and (16) the distributions of pressure and vorticity are expressed as

$$p - i\mu\Omega = \frac{i\mu K_1}{2}\left[z + k(z)\right]\left[1 + \frac{z - 2s_1}{k(z)}\right] - \frac{i\mu K_2}{2}\left[z - k(z)\right]\left[1 - \frac{z - 2s_2}{k(z)}\right] \quad . \tag{23}$$

On the plates, shear stress τ is equal to $-\mu\Omega$ since v vanishes for any x. Hence the distributions of pressure and shear stress on the right plate can be obtained by calculating the boundary values of (23). The results are written as

$$p(x, \pm 0) = 0 \quad , \tag{24}$$

$$\begin{aligned} \tau(x, \pm 0) &= \frac{K_1}{2}\left(x \pm \sqrt{x^2 - 1}\right)\left[\left(1 \pm \frac{x}{\sqrt{x^2 - 1}}\right) \mp \frac{2s_1}{\sqrt{x^2 - 1}}\right] \\ &\quad - \frac{K_2}{2}\left(x \mp \sqrt{x^2 - 1}\right)\left[\left(1 \mp \frac{x}{\sqrt{x^2 - 1}}\right) \pm \frac{2s_2}{\sqrt{x^2 - 1}}\right] \\ &= K_1\left[f_1(x) + s_1 g_1(x)\right] + K_2\left[f_2(x) + s_2 g_2(x)\right] \quad . \end{aligned} \tag{25}$$

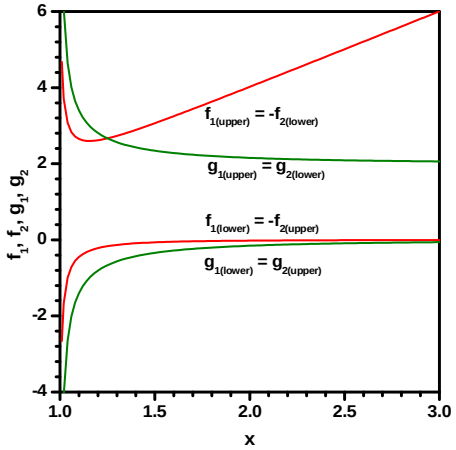


Fig. 6. The variations of f_1, f_2, g_1, and g_2 with respect to x>1.

A stagnation points, if any, on the plate is found by searching for a point at which shear stress changes signs. As is given in (25), the shear stress distribution on the right plate is a linearfunction of the strengths and centerline shifts. The variations of f_1, f_2, g_1, and g_2 with respect to x are plotted in Fig. 6. For large x, the function f_1 on the upper side ($-f_2$ on the lower side)

increases with x. But f_1 on the lower side ($-f_2$ on the upper side), and g_1 and g_2 asymptote to a finite value or zero rather slowly with the increase of x.

CONCLUSION

In this study, two-dimensional viscous impinging flow is investigated based on the Stokes approximation, arising when a fluid stagnates toward a slit of a flat plate from both sides of it. The flow field including the streamline patterns are determined by the use of complex functions. For any stagnation flow strengths and centerline shifts, flowrate through the slit is zero and there exists no viscous eddy. The stagnation point either lies on the slit or on the sides of the plate. For a complete understanding of the behaviors of stagnation flow over a flat plate with a crack, further studies are needed, such as on the effect of the nonlinearity or the pressure difference.

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