SEMI-EXACT ALGEBRAIC EXPRESSIONS FOR THE NATURAL FREQUENCIES OF RECTANGULAR KIRCHHOFF-LOVE PLATES

MICHAEL J. LEAMY

George W. Woodruff School of Mechanical Engineering,
Georgia Institute of Technology, Atlanta, GA 30332-0405

Abstract— Semi-exact, closed-form algebraic expressions are developed for the natural frequencies of rectangular Kirchhoff-Love plates. The approach employs plane waves, their edge reflections, and phase closure considerations. The semi-exact nature is such that the analysis exactly satisfies plate boundary conditions along each edge when taken in isolation, but not fully when combined, and thus is approximate near a corner. As frequency increases, the expressions become increasingly more accurate. For clamped square plates, closed-form expressions are developed in algebraic form for the first time and it is shown that the expressions for the first twenty non-dimensional natural frequencies are within 0.06% of their exact values.

I. INTRODUCTION

The study of the free response of thin, rectangular plates has an extensive history dating back to the derivation of the field equations by Love in 1888 [1] using kinematic assumptions first proposed by Kirchhoff in 1850 [2]. The formulation is reviewed in standard texts such as those by Timoshenko and Woinowsky-Krieger [3], Szilard [4], and Reddy [5]. Rectangular plate solutions for free response - i.e., natural frequencies (eigenvalues) and mode shapes (eigenfunctions) - have received numerous treatments, particularly when the boundary conditions do not include two edges simply supported. Exact solutions can be found in closed form when the rectangular plate has two opposite faces simply supported. When two opposing edges are not simply supported, exact solutions have traditionally been unavailable and instead either numerical (finite element, finite difference, etc.) approaches have been employed, or approximate analytical solutions have been sought using techniques such as the Rayleigh and Ritz methods [6, 7]. The approximate analytical solutions are only as good as their assumed basis functions, which concerned many of the studies employing these techniques (e.g., see works by Warburton [8] using beam eigenfunctions and by Janich [9] using trigonometric functions). Very recently, Xing and Liu [10] presented equations for exact rectangular plate natural frequencies and mode shapes by recasting the field equations in Hamiltonian dual form and solving them using a classical separation of variables approach. Their approach yields correct solutions for previously intractable problems, such as fully clamped plates. This is a very important development; however, it does not fully close the discussion of these problems as the eigenvalue solutions are not in closed form. Instead, they require the simultaneous solution of three nonlinear equations, two of which are transcendental. This is problematic in two respects. First, the solution approach requires an iterative, numerical scheme (such as Newton-Raphson) to find the roots and thus the solutions are not easily archived in general treatments (engineering handbooks, field manuals, etc.). Second, numerical nonlinear root finding can potentially misidentify the first root (i.e., the fundamental natural frequency) or miss roots, particularly in cases where two roots are closely spaced, or where a single root is isolated between two groups of clustered roots.

In what follows, a phase closure approach is developed for finding semi-exact, closed-form expressions for the natural frequencies of thin rectangular plates. The chief advantage of the presented approach over existing approaches is the development of highly-accurate, closed-form algebraic expressions suitable for archiving in compilations such as handbooks and manuals. This is in contrast to the only available exact approach and its need to numerically solve three nonlinear equations, two of which are transcendental. Only rectangular plates with common sets of boundary conditions (free, simply-supported, and clamped) are treated in detail, but it is believed the presented approach should generalize to other plate shapes and boundary conditions, such as triangular or hexagonal plates, with edges restrained by flexible supports of non-negligible mass and dissipation.

Solution Approach

The equation of motion governing the free response of thin, linear, isotropic, Kirchhoff-Love plates is well-known [11] and given by,

$$D\nabla^4 w(x,y,t) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0, \qquad (1)$$

where $w(x,y,t)$ denotes the plate's transverse displacement, D the plate bending rigidity, ρ the density (mass per unit volume), h the (small) plate thickness, and ∇^4 denotes the biharmonic differential operator, which in rectangular Cartesian coordinates is given by $\nabla^4 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial y^2 \partial x^2} + \frac{\partial^4}{\partial y^4}\right)$. In terms of linear material properties, the bending rigidity can be expressed as $D = \frac{Eh^3}{12(1-\nu^2)}$ where E denotes Young's Modulus and ν denotes Poisson's ratio. While it is common to seek separable solutions to Eq. (1) leading to functional dependence on sines, cosines, and their hyperbolic variants, this work instead starts with propagating and evanescent plane wave solutions. Graff [11] provides a complete discussion of these solutions together with further discussion on their reflections from common interfaces.

Figure 1 provides the geometry necessary to study plane wave solutions and their reflections from a common support (e.g., a clamped edge). An incident plane wave **A** with frequency ω travels in the positive x-direction with wavenumber component ξ, and in the y-direction with wavenumber component $-\zeta_2$. At the supported edge, the incident wave is reflected into an outgoing propagating wave **B** and/or a propagating evanescent wave **C**. As detailed in Graff [11], the total field for such a situation (and others) is a solution to Eq. (1) and is given by,

$$w(x,y,t) = Ae^{i(\xi x - \zeta_2 y - \omega t)} + Be^{i(\xi x + \zeta_2 y - \omega t)} + Ce^{-\zeta_1 y}e^{i(\xi x - \omega t)} + De^{\zeta_1 y}e^{i(\xi x - \omega t)}. \quad (2)$$

Note that the **D** wave corresponding to the last term in Eq. (2) leads to exponential growth in the y-direction and therefore its amplitude D is set to zero. As evident in Eq. (2), the **C** wave propagates along the support edge and decays in the y-direction with exponential dependence ζ_1. Substitution of this solution form into the plate governing equation yields dispersion information,

$$\omega = \sqrt{\frac{D}{\rho h}}\gamma^2, \qquad \gamma^2 = \xi^2 + {\zeta_2}^2, \qquad \zeta_1 = \sqrt{\xi^2 + \gamma^2}, \quad (3)$$

where γ denotes the resultant (or total) wavenumber.

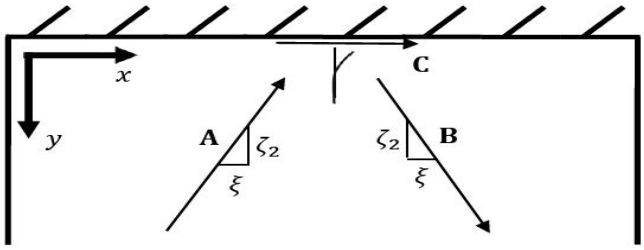


Figure 1: Wave reflection and conversion at a supported plate edge (e.g., clamped).

The solution form given by Eqs. (2) – (3), together with boundary conditions, allows one to develop a relationship for waves **B** and **C** dependent on the incident wave **A** [11]. For a simply supported edge ($w(x,0,t) = \partial^2 w(x,0,t)/\partial y^2 + \nu\,\partial^2 w(x,0,t)/\partial x^2 = 0$),

$$B = -A, \quad C = 0. \quad (4)$$

Similarly, for a clamped edge ($w(x,0,t) = \partial w(x,0,t)/\partial y = 0$),

$$B = -\frac{\zeta_1 - i\zeta_2}{\zeta_1 + i\zeta_2}A, \quad C = -\frac{2i\zeta_2}{\zeta_1 + i\zeta_2}A. \quad (5)$$

The solution form Eq. (2), and knowledge of wave reflection from the edge supports, is sufficient background to determine the vibration modes. Note that since the spatial operator in Eq. (1) is symmetric, analogous to symmetric receptances in periodic systems [12, 13], it can be expected that natural frequencies will fall on passband dispersion branches. Thus the plate's natural frequencies can be found by tracing propagating wave paths as they reflect within the plate, quantifying the phase shift due to propagation distance and reflections, and then enforcing phase closure on ξ, ζ_2, and γ. Closure on all three requires elaboration of wave directions which result in periodic wave paths. For rectangular plates with lengths in the x- and y-directions of extent a and b, these are paths in which the ratio of nondimensional quantities ξa and $\zeta_2 b$ forms a rational quotient q/p, p and q being integers.

We first consider a square plate whose sides have length a. All modes can be determined by considering waves obeying $q\zeta_2 = p\xi$ where q and p share no common divisors. The closed phase paths for two such cases are illustrated in Fig. 2. These paths illustrate cases where $q < p$; however, the expressions developed hold when $q > p$. Note that for $q = 2, p = 3$ and $q = 4, p = 5$, the phase paths originating at the midpoint of the left edge reflect from a corner, as indicated by dashed phase paths. To avoid ambiguity associated with such a reflection, the phase paths are considered to originate just below the midpoint, as shown in Figs. 2a and 2c. The three phase closure relationships are given by,

$$2qa\xi + 2pR_{1x} + 2qR_{2x} = 2m\pi, \quad (6)$$

$$2pa\zeta_2 + 2pR_{1y} + 2qR_{2y} = 2n\pi, \quad (7)$$

$$2\sqrt{p^2+q^2}a\gamma + 2pR_1 + 2qR_2 = 2l\pi. \quad (8)$$

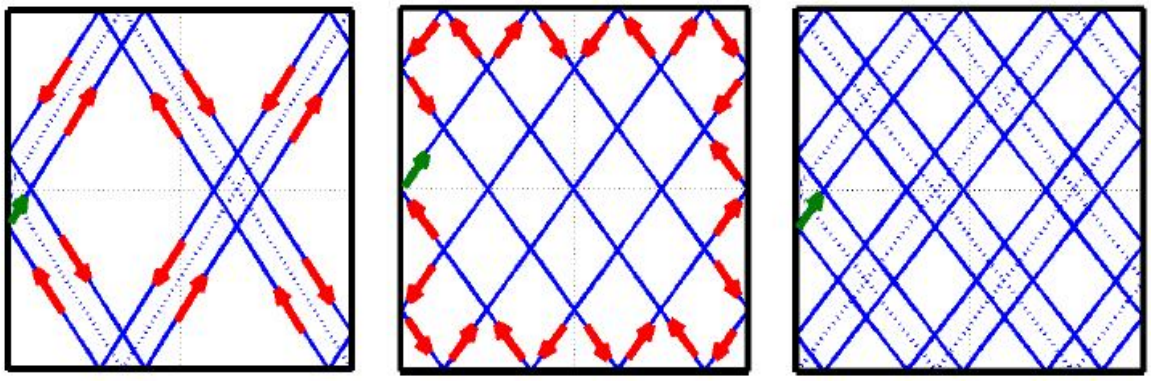


Figure 2: Wave paths and reflections associated with waves characterized by $q\zeta_2 = p\xi$ where (left) $q = 2, p = 3$, (middle) $q = 3, p = 4$, and (right) $q = 4, p = 5$.

The first equation states that the propagation change over the x-distance traveled ($2a$), plus the x-components of reflected phase change R_1 and R_2, must sum to an integer multiplier m of 2π. This is a statement of phase closure in the x-direction. The second equation states a similar phase closure for propagation in the y-direction with integer n introduced, and the third states phase closure in the direction of wave propagation with integer l introduced. For $\zeta_2 \neq \xi$, that incidence angles for reflections from x-edges will be different from y-edges, and thus these reflections are distinguished by subscripts 1 and 2. Note that reflected phase change R is the complex phase associated with the multipliers of A in the expressions for B in Eqs. (4)–(5). The amount of this phase change correctly proportioned to the x- and y-directions (R_x and R_y, respectively) must still be determined.

Solving the three phase closure relationships (further details provided in [14]), it is found that the relationship between n and m must satisfy $n = p^2/q^2 m$, which does not guarantee integer values of n. This suggests a change to a new integer $m' = m/q^2$ such that $n = p^2 m'$. Proceeding with this change and renaming m' to m for notational convenience, one finds,

$$l = q^2 m + n,\ n = p^2 m,\ \xi = \frac{q(l\pi - pR_1 - qR_2)}{(p^2+q^2)a},\ \gamma = \frac{\sqrt{p^2+q^2}}{q}\xi,\ R_{1x} = \frac{q^2}{p^2+q^2}R_1,\ R_{2x} = \frac{q^2}{p^2+q^2}R_2. \quad (9)$$

The natural frequencies are subsequently given as,

$$\omega_{qm,pm} = (p^2+q^2)\left(\frac{(p^2+q^2)m\pi - pR_{1\,q,p} - qR_{2\,q,p} + (p+q)\pi}{(p^2+q^2)a}\right)^2 \sqrt{\frac{D}{\rho h}},\ q = 1,2,3,\ldots,\ p = 1,2,3,\ldots,\ m = 1,2,3,\ldots \quad (10)$$

Note that this expression holds regardless of the edge support conditions. It is also invariant to exchanging p and q, as expected, thus giving rise to repeated roots. Each support condition provides different values for $R_{1\,q,p}$ and $R_{2\,q,p}$, where each is known *a priori* from knowledge of p and q alone. For the case of a simply-supported edge, these reflections are always equal to π. For a clamped edge, they follow from Eqs. (3) and (5). Using $\zeta_1 = \sqrt{\xi^2+\gamma^2}$, $\gamma^2 = \xi^2 + {\zeta_2}^2$, and $\zeta_2 = {}^{p}/_{q}\,\xi$,

$$R_{1\,q,p}^{C} = \arg\left(-\frac{\zeta_1 - i\zeta_2}{\zeta_1 + i\zeta_2}\right) = \arg\left(-\frac{\sqrt{2q^2+p^2} - ip}{\sqrt{2q^2+p^2} + ip}\right) = \pi - \operatorname{atan}\left(\frac{p}{q^2}\sqrt{2q^2+p^2}\right). \quad (11)$$

The expression for $R_{2\,q,p}^{C}$ is found similarly with the opposite ratio $\zeta_2 = {}^{q}/_{p}\,\xi$,

$$R_{2\,q,p}^{C} = \arg\left(-\frac{\sqrt{2p^2+q^2} - iq}{\sqrt{2p^2+q^2} + iq}\right) = \pi - \operatorname{atan}\left(\frac{q}{p^2}\sqrt{2p^2+q^2}\right). \quad (12)$$

Rectangular plates with sides of length a and b are considered next for the general case of wavenumber components related by $qb\zeta_2 = pa\xi$. In effect, this relationship introduces dimensionless wavenumber components $\bar{\zeta}_2 = b\zeta_2$ and $\bar{\xi} = a\xi$ such that $q\bar{\zeta}_2 = p\bar{\xi}$. Although not presented herein, closed-form expressions can also be found using similar arguments employed for square plates. See [14] for further details.

SQUARE PLATE RESULTS

For square plates with four clamped edges, Table 1 presents the first 20 unique natural frequencies and compares them to approximate results reported by Leissa [7] and exact results computed numerically using the Hamiltonian dual method introduced by Xing and Liu [10]. The semi-exact algebraic expressions developed herein compare very favorably to the numerical roots found using the exact equations. The largest absolute error in the first 20 presented natural frequencies is less than 0.06%. In fact, nearly all of the absolute errors are under 0.01%. A trend is clear in the semi-exact solutions that as frequency increases, so does the accuracy of the frequencies.

CONCLUDING REMARKS

Using a phase closure argument on plane waves, this article detailed a semi-exact approach for finding algebraic expressions for the natural frequencies of rectangular Kirchhoff-Love plates. It has been shown that these expressions yield very high accuracy when compared to other known approaches, with the advantage of being reported in closed-form.

ACKNOWLEDGMENTS

The author would like to thank the National Science Foundation for support of this research under Grant No. (CMMI 1332862).

REFERENCES

[1] Love, A. E. H., 1888, "The Small Free Vibrations and Deformation of a Thin Elastic Shell," Philosophical Transactions of the Royal Society of London. A, **179**, pp. 491-546.
[2] Kirchhoff, G. R., 1850, "Uber Das Gleichgewicht Und Die Bewegung Einer Elastischen Scheibe," Journal für die reine und angewandte Mathematik, **40**, pp. 51-88.
[3] Timoshenko, S., Woinowsky-Krieger, S., and Woinowsky-Krieger, S., 1959, *Theory of Plates and Shells*, McGraw-Hill, New York.
[4] Szilard, R., 1974, *Theory and Analysis of Plates: Classical and Numerical Methods*, Prentice-Hall, Englewood Cliffs, NJ.
[5] Reddy, J. N., 2006, *Theory and Analysis of Elastic Plates and Shells*, CRC press, Boca Raton, FL.
[6] Leissa, A. W., 1973, "Free Vibration of Rectangular-Plates," Journal of Sound and Vibration, **31**(3), pp. 257-293.
[7] Leissa, A. W., 1969, "Vibration of Plates," Technical Report No. National Aeronautics and Space Administration, Washingtion, D.C.
[8] Warburton, G., 1954, "The Vibration of Rectangular Plates," Proceedings of the Institution of Mechanical Engineers, **168**(1), pp. 371-384.
[9] Janich, R., 1962, "Die Näherungsweise Berechnung Der Eigenfrequenzen Von Rechteckigen Platten Bei Verschiedenen Randbedingungen," Die Bautechnik, **3**, pp. 93-99.
[10] Xing, Y., and Liu, B., 2009, "New Exact Solutions for Free Vibrations of Rectangular Thin Plates by Symplectic Dual Method," Acta Mechanica Sinica, **25**(2), pp. 265-270.
[11] Graff, K. F., 1975, *Wave Motion in Elastic Solids*, *Oxford Engineering Science Series*, Clarendon Press, Oxford.
[12] Mead, D. J., 1975, "Wave-Propagation and Natural Modes in Periodic Systems .1. Mono-Coupled Systems," Journal of Sound and Vibration, **40**(1), pp. 1-18.
[13] Mead, D. J., 1975, "Wave-Propagation and Natural Modes in Periodic Systems .2. Multi-Coupled Systems, with and without Damping," Journal of Sound and Vibration, **40**(1), pp. 19-39.
[14] Leamy, M. J., 2016, "Semi-Exact Natural Frequencies for Kirchhoff-Love Plates Using Wave-Based Phase Closure," Journal of Vibration and Acoustics, in press.

Table 1: Nondimensional natural frequencies for a CCCC square plate with comparisons to results cataloged by Leissa [7] and computed numerically using exact expressions [10].

#	*m*	*q*	*p*	*(mq,mp)*	*Leissa [7]* $\omega a^2\sqrt{\rho h/D}$	*Exact [10]* $\omega a^2\sqrt{\rho h/D}$	*Present* $\omega a^2\sqrt{\rho h/D}$
1	1	1	1	(1,1)	35.10	35.1124	35.09193
2	1	1	2	(1,2)	72.90	72.8994	72.88823
3	2	1	1	(2,2)	107.47	107.4689	107.4690
4	1	1	3	(1,3)	131.63	131.6289	131.6154
5	1	2	3	(2,3)	164.39	164.3867	164.3798
6	1	1	4	(1,4)	210.35	210.3617	210.3520
7	3	1	1	(3,3)	219.32	219.3245	219.3245
8	2	1	2	(2,4)	242.20	242.1969	242.1844
9	1	3	4	(3,4)	295.69	295.6981	295.6943
10	1	1	5	(1,5)		308.9291	308.9221
11	1	2	5	(2,5)		340.2442	340.2308
12	4	1	1	(4,4)	370.66	370.6585	370.6585
13	1	3	5	(3,5)		392.7956	392.7866
14	1	1	6	(1,6)		427.2802	427.2750
15	2	1	3	(2,6)		458.2706	458.2581
16	1	4	5	(4,5)		466.6129	466.6105
17	3	1	2	(3,6)		510.1887	510.1766
18	5	1	1	(5,5)		561.4708	561.4708
19	1	1	7	(1,7)		565.3951	565.3911
20	2	2	3	(4,6)		583.1441	583.1376

★★★