BENDING ANALYSIS OF THICK LAMINATED PLATES BY A BOUNDARY ELEMENT METHOD

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Abstract— In this paper, a Boundary Element Method (BEM) is developed for analysis of moderately thick laminated plates modeled by Mindlin's theory. Laminated plates are composed of orthotropic layers where shear deformation is considered. The governing equations are three coupled linear partial differential equations of second order with three physical conditions along the plate boundary. The present method is achieved using the concept of the analog equation method (AEM). According to this principle, the original governing equations are replaced by three uncouple Poisson's equations with fictitious sources under the same boundary conditions. The fictitious sources are established using a technique based on the BEM and approximated by radial basis functions series. The present method has the advantages of the BEM in a sense that the discretization and integration are performed only on the boundary. To validate the effectiveness, accuracy as well as the applicability of the proposed method, numerical results of plate problems are presented.

Keywords— Thick plates, Mindlin plates, Laminated Plates, Boundary Element Method, Analog Equation method.

1. INTRODUCTION

Composite laminates are formed by stacking layer of different composite materials and/or fiber orientation. Laminated plates consisting of various layers of different mechanical properties gives the designers the opportunity to increase the strength, stiffness, minimize the weight and thus optimize the design according to the structural requirement.

Analyses of composite laminated plates have been primarily based on two approaches: (1) Equivalent single-layer theories (2-D) using classical laminated plate theory or shear deformation laminated plate theory. (2) Three-dimensional elasticity theory (3-D) using traditional 3-D elasticity formulations or layerwise theories. The equivalent single layer (ESL) plate theories are derived from the 3-D elasticity theory by making suitable assumptions concerning the kinematics of deformation or the stress state through the thickness of the laminated plates. These assumptions allow the reduction of a 3-D problem to a 2-D problem. On the other hand, each layer is modeled as a 3-D solid in the 3-D elasticity formulations or in layerwise theories.

As the plate thickness increases, the assumptions of the Kirchhoff theory or classical plate theory are violated and shear deformation plate theories are needed. Mindlin [1] was among the first who presented the plate theory which take into account the shear deformation of moderately thick plates. Mindlin's theory can be considered as the first-order shear deformation theory (FSDT).

Analytical solutions of thick laminated plates have been presented only for simple boundary conditions, plate shapes, and external loading [2-3]. However, solution for realistic plate problems can be obtained by numerical methods such as finite element method (FEM) [4]. J.N. Reddy and W.C. Chao [5] developed exact closed-form for both of anisotropic and laminate rectangular thick plate. A finite element model based on the YNS theory was also computed. More recently, the boundary element method (BEM) has been proven to be a powerful solution for engineering problems. However, a few work of BEM for analyzing laminated plate problems have been reported due to their mathematical difficulty. Static analysis of thick layered anisotropic plates with BEM has been proposed by Baboukos and Katsikadelis [6]. In their formulations, the domain integrals are approximated by using linear triangular elements.

In this research we study a bending problem of moderately thick laminated plates modeled by Mindlin's plate theory. The boundary element method is developed in conjunction with the concept of Analog Equation Method (AEM) [7]. The governing equations are three coupled linear partial differential equations of second order with three physical conditions along the plate boundary and written in terms of the transverse deflection of the middle surface and it rotations about the x and y axes. According to the principle of AEM, the original governing equations are replaced by three uncouple Poisson's equations with fictitious sources under the same boundary conditions. The fictitious sources are established using a technique based on the BEM and approximated by radial basis functions series. The present method has the advantages of the BEM in a sense that the discretization and integration are performed only on the boundary. No limitations on boundary conditions, plate shapes, and external applied loads are presented in the proposed formulations. Numerical results of thick laminated plate problems are given to demonstrate the accuracy and applicability of the present method.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

Consider a plate of total thickness h composed of N orthotropic layers occupying the two-dimensional multiply connected domain Ω of the xy -plane with the boundary $\Gamma \cup_{i=0}^{i=K} \Gamma_i$ (see Fig. 1). The curve Γ_i(i=0,1,2,…k) may be piecewise smooth, that is, the boundary may have a finite number of corners. The xy-plane of the problem is taken in the undeformed midplane of the laminated plate. The z- axis is taken positive downward from the midplane. The m th layer is located between the points $z = z_m$ and $z = z_{m+1}$ in the thickness direction

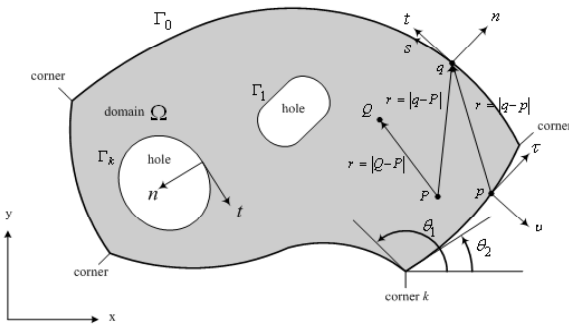


Fig. 1

The first-order shear deformation laminated plate theory with Mindlin's assumption for the shear deformations is applied. The displacement field is expressed by the relations

$$u(x,y,z) = z\phi_x\,(x,y) \tag{1a}$$

$$v(x,y,z) = z\phi_y(x,y) \tag{1b}$$

$$w(x,y,z) = w(x,y) \tag{1c}$$

where w denote the displacement components in the z axis of a point on the midplane, and the functions $\phi_x, -\phi_y$ represent the rotations about the y and x axes, respectively

We assume that each layer of a laminate is orthotropic and obey Hooke's law. The linear constitutive relations for the m th orthotropic lamina are

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(m)} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^{(m)} \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \tag{2a}$$

$$\begin{Bmatrix}\sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}^{(m)} = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}^{(m)} \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{2b}$$

$$\sigma_{zz} = 0 \tag{2c}$$

where $\bar{Q}_{ij}^{(m)}$ are the known stiffness in terms of the engineering constants of the m th layer with respect to the laminate coordinates (x,y,z).

In composite laminated beams and plates, the transverse shear stresses vary at least quadratically through the layer thickness. A remedy to cope with this discrepancy between the actual stress state and the constant stress state predicted by the first-order theory is to modify the transverse shear stiffness of the plate by introducing a shear correction factor K_s.

The stress resultants on a cross-section are expressed by forces and moments per unit length along it and related to the internal stresses by the equations

$$\begin{Bmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{Bmatrix} = \sum_{m=1}^{N}\int_{z_m}^{z_{m+1}} z\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} dz \tag{3a}$$

$$\begin{Bmatrix}Q_y\\ Q_x\end{Bmatrix} = K_s\sum_{m=1}^{N}\int_{z_m}^{z_{m+1}} \begin{Bmatrix}\sigma_{yz}\\ \sigma_{xz}\end{Bmatrix} dz \tag{3b}$$

where M_{xx}, M_{yy} are the bending moments, M_{xy} is the twisting moment, and Q_x, Q_y are the shear forces.

We have the following laminate constitutive equations:

$$\begin{Bmatrix}M_{xx}\\ M_{yy}\\ M_{xy}\end{Bmatrix} = \begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix} \begin{Bmatrix}\frac{\partial\phi_x}{\partial x}\\ \frac{\partial\phi_y}{\partial y}\\ \frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\end{Bmatrix} \tag{4a}$$

$$\begin{Bmatrix}Q_y\\ Q_x\end{Bmatrix} = K_s\begin{bmatrix}A_{44} & A_{45}\\ A_{45} & A_{55}\end{bmatrix} \begin{Bmatrix}\phi_y+\frac{\partial w}{\partial y}\\ \phi_x+\frac{\partial w}{\partial x}\end{Bmatrix} \tag{4b}$$

where the stiffnesses A_{ij}, B_{ij}, D_{ij} are defined in terms of the lamina stiffnesses $\bar{Q}_{ij}^{(m)}$ as

$$A_{ij} = \sum_{m=1}^{N}\bar{Q}_{ij}^{(m)}(z_{m+1}-z_m) \tag{5a}$$

$$D_{ij} = \tfrac{1}{3}\sum_{m=1}^{N}\bar{Q}_{ij}^{(m)}(z_{m+1}^3-z_m^3) \tag{5b}$$

The governing differential equations of the laminate plate subjected to a distributed transverse load $q(x,y)$ can be expressed in terms of functions (w,ϕ_x,ϕ_y) as

$$\frac{\partial}{\partial x}\left[K_sA_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)+K_sA_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)\right]+$$

$$\frac{\partial}{\partial y}\left[K_sA_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)+K_sA_{45}\left(\frac{\partial w}{\partial x}+\phi_x\right)\right]+q(x,y)=0 \tag{6a}$$

$$\frac{\partial}{\partial x}\left[D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}+D_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]+$$
$$\frac{\partial}{\partial y}\left[D_{16}\frac{\partial\phi_x}{\partial x}+D_{26}\frac{\partial\phi_y}{\partial y}+D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]-$$

$$K_s A_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)\text{-}K_s A_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \qquad (6b)$$

$$\frac{\partial}{\partial x}\left[D_{16}\frac{\partial \phi_x}{\partial x}+D_{26}\frac{\partial \phi_y}{\partial y}+D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]+$$
$$\frac{\partial}{\partial y}\left[D_{12}\frac{\partial \phi_x}{\partial x}+D_{22}\frac{\partial \phi_y}{\partial y}+D_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right]-$$

$$K_s A_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)\text{-}K_s A_{45}\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \qquad (6c)$$

Furthermore, the deflection w must satisfy the following boundary conditions on the boundary:

$$\alpha_1 w+\alpha_2 Q_n=\alpha_3 \qquad (7a)$$

$$\beta_1\phi_n+\beta_2 M_n=\beta_3 \qquad (7b)$$

$$\gamma_1\phi_t+\gamma_2 M_{nt}=\gamma_3 \qquad (7c)$$

where $\alpha_i=\alpha_i(c)$, $\beta_i=\beta_i(c)$, $\gamma_i=\gamma_i(c)$, $c(x,y)\in\Gamma$ (i=1,2,3) are given functions specified on the boundary Γ; Q_n, M_n and M_{nt} represent respectively the shear force, bending moment and twisting moment along the boundary and ϕ_n and $-\phi_t$ represent the angles at the midplane of the plate with respect to the axes t and n, respectively (see Fig. 1). The boundary conditions (7a,b,c) are the most general linear boundary conditions for the moderately thick plate problem, including also transverse and rotational elastic supports. All types of conventional boundary can be derived from equations (7a,b,c) by specifying appropriately the function α_i, β_i and γ_i [8].

3. THE AEM SOLUTION AS A BOUNDARY-ONLY METHOD

The boundary value problem described by equations (6)-(7) is solved using the analog equation method (AEM). This method is applied to the problem at hand as follows.

Let $w=w(x,y), \phi_x=\phi_x(x,y)$ and $\phi_y=\phi_y(x,y)$

be the sought solution of equations (6)-(7). These functions are twice differentiable in the domain Ω. Thus, if the Laplacian operator is applied to them, we have

$$\nabla^2 w=b^{(1)}(x,y), \nabla^2\phi_x=b^{(2)}(x,y), \nabla^2\phi_y=b^{(3)}(x,y) \qquad (8a,b,c)$$

Equations (8a,b,c) are the analog equations of the problem at hand. They indicate that the solution of equations (6a,b,c) could be established by solving these Poisson's equations under the boundary conditions (7a,b,c), if the fictitious sources $b^{(l)}(x,y)(l=1,2,3)$ were known. They can be solved by using the BEM. Thus, the solution is obtained in integral forms as

$$\varepsilon w(p)=-\int_\Gamma\left(u^* w_{,n}-u^*_{,n}\, w\right)ds+\int_\Omega\left(u^* b^{(1)}\right)d\Omega \qquad (9)$$

$$\varepsilon\phi_x(p)=-\int_\Gamma\left(u^*\phi_{x,n}-u^*_{,n}\,\phi_x\right)ds+\int_\Omega\left(u^* b^{(2)}\right)d\Omega \qquad (10)$$

$$\phi_y(p)=-\int_\Gamma\left(u^*\phi_{y,n}-u^*_{,n}\,\phi_y\right)ds+\int_\Omega\left(u^* b^{(3)}\right)d\Omega \qquad (11)$$

where $u^*=lnr/2\pi$ is the fundamental solution of the Laplace equation and $u^*_{,n}$ is its derivative normal to the boundary at the field point q with $r=|q-p|$ (see Fig. 2); ε is the free term coefficient, which takes the value $\varepsilon=1$ if $p\in\Omega$ and $\varepsilon=\alpha/2\pi$ if $p\in\Gamma$; α is the interior angle between the tangents of the boundary at point p. Note that $\varepsilon=1/2$ for points where the boundary is smooth.

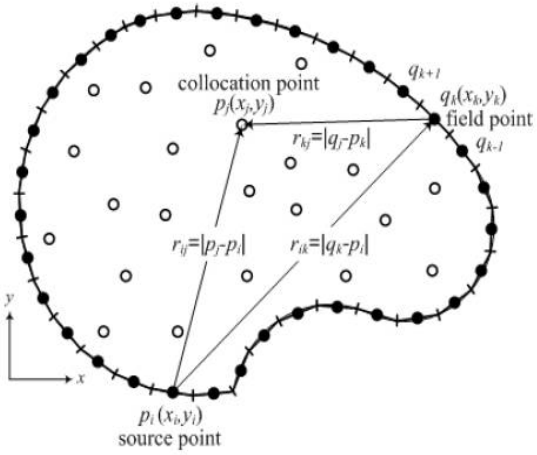


Fig. 2

Equations (9)-(11) are domain-boundary integral since they include the unknown fictitious sources in the domain integrals. They could be solved using domain discretization for the domain integrals [6]. However, in order that the method maintains its pure boundary character, the domain integrals in equations (9)-(11) are converted to boundary line integrals and approximating fictitious sources using radial basis functions. Thus we have

$$b^{(l)}=\sum_{j=1}^{M} a_j^{(l)} f_j, l=1,2,3 \quad , \qquad (12)$$

where $f_j=f_j(r)$ are M radial-basis approximation functions centered at M collocation points $p_j(x_j,y_j)$ and $a_j^{(l)}$ are M (in total 3M) coefficients to be determined. Note that $r\equiv r_{ji}=|p_i-p_j|$ is the

distance between the collocation point $p_j(x_j, y_j)$ and any point $p_j(x_j, y_j) \in \Omega \cup \Gamma$ (see Fig. 2).

By substituting equation (12) into equations (9)-(11), the solution of equations (6a,b,c) can be rewritten in the integral form as

$$\varepsilon w(p) = -\int_\Gamma \left(u^* w_{,n} - u^*_{,n} w\right) ds + \sum_{j=1}^{M} a_j^{(1)} \left\{ \int_\Omega \left(u^* f_j\right) d\Omega \right\}, \tag{13}$$

$$\varepsilon \phi_x(p) = -\int_\Gamma \left(u^* \phi_{x,n} - u^*_{,n} \phi_x\right) ds + \sum_{j=1}^{M} a_j^{(2)} \left\{ \int_\Omega \left(u^* f_j\right) d\Omega \right\}, \tag{14}$$

$$\varepsilon \phi_y(p) = -\int_\Gamma \left(u^* \phi_{y,n} - u^*_{,n} \phi_y\right) ds + \sum_{j=1}^{M} a_j^{(3)} \left\{ \int_\Omega \left(u^* f_j\right) d\Omega \right\}, \tag{15}$$

If we define the function $\hat{u}_j = \hat{u}_j(r_{ij}) = \hat{u}_j(x, y)$ as a particular solution of

$$\nabla^2 \hat{u}_j = f_j \quad , \tag{16}$$

and use the Green's second identity, one obtains

$$\int_\Omega u^* f_j d\Omega = \int_\Omega \left(u^* \nabla^2 \hat{u}_j\right) d\Omega = \int_\Omega \left(\hat{u}_j \nabla^2 u^*\right) d\Omega + \int_\Gamma u^* \hat{u}_{j,n} - u^*_{,n} \hat{u}_j \tag{17}$$

which is substituted in equations (13)-(15) to yield

$$\varepsilon w(p) = -\int_\Gamma \left(u^* w_{,n} - u^*_{,n} w\right) ds + \sum_{j=1}^{M} a_j^{(1)} \left\{ \varepsilon \hat{u}_j(p) + \int_\Gamma \left(u^* \hat{u}_{j,n} - u^*_{,n} \hat{u}_j\right) ds \right\}, \tag{18}$$

$$\varepsilon \phi_x(p) = -\int_\Gamma \left(u^* \phi_{x,n} - u^*_{,n} \phi_x\right) ds + \sum_{j=1}^{M} a_j^{(2)} \left\{ \varepsilon \hat{u}_j(p) + \int_\Gamma \left(u^* \hat{u}_{j,n} - u^*_{,n} \hat{u}_j\right) ds \right\}, \tag{19}$$

$$\varepsilon \phi_y(p) = -\int_\Gamma \left(u^* \phi_{y,n} - u^*_{,n} \phi_x\right) ds + \sum_{j=1}^{M} a_j^{(3)} \left\{ \varepsilon \hat{u}_j(p) + \int_\Gamma \left(u^* \hat{u}_{j,n} - u^*_{,n} \hat{u}_j\right) ds \right\}, \tag{20}$$

Note that a particular solution of equation (16), $\hat{u}_j$, can be always established if f_j is specified.

The additional two integral equations are obtain from equations (19) and (20) by first differentiating with respect to a direction t and then finding the limit as $p \in \Omega \to p \in \Gamma$ choosing t to coincide with the tangent $\tau = \{\tau_x, \tau_y\}$ to the boundary at point $p \in \Gamma$. The limiting procedure is similar to that of the BEM book [7]. For point p where the boundary is smooth, we find

$$\frac{1}{2}\phi_{x,\Gamma}(p) = -\int_\Gamma \left(u^*_{,\Gamma} \phi_{x,n} - u^*_{,n\tau}[\phi_x - \phi_x(p)]\right) ds + \sum_{j=1}^{M} a_j^{(2)} \left\{ \frac{1}{2} u^*_{,\Gamma}(p) + \int_\Gamma \left(u^*_{,\Gamma} \hat{u}_{j,n} - u^*_{,n\tau}[\hat{u}_j - \hat{u}_j(p)]\right) \right\} \tag{21}$$

$$\frac{1}{2}\phi_{y,\Gamma}(p) = -\int_\Gamma \left(u^*_{,\Gamma} \phi_{y,n} - u^*_{,n\tau}[\phi_y - \phi_y(p)]\right) ds + \sum_{j=1}^{M} a_j^{(3)} \left\{ \frac{1}{2} u^*_{,\Gamma}(p) + \int_\Gamma \left(u^*_{,\Gamma} \hat{u}_{j,n} - u^*_{,n\tau}[\hat{u}_j - \hat{u}_j(p)]\right) \right\} \tag{22}$$

The derivatives of the deflection w, rotations ϕ_x and ϕ_y inside the domain Ω are obtained by direct differentiation of equations (18) - (20) respectively. Thus, for the sake of conciseness we can write the integral representation of the deflection w, rotations ϕ_x and ϕ_y, and their derivatives up to second order as

$$w_{,vg}(P) = -\int_\Gamma \left(u^*_{,vg} w_{,n} - u^*_{,nvg} w\right) ds + \sum_{j=1}^{M} a_j^{(1)} \left\{ \hat{u}_{j,vg}(P) + \int_\Gamma \left(u^*_{,vg} \hat{u}_{j,n} - u^*_{,nvg} \hat{u}_j\right) ds \right\}, \tag{23}$$

$$\phi_{x,vg}(P) = -\int_\Gamma \left(u^*_{,vg} \phi_{x,n} - u^*_{,nvg} \phi_x\right) ds + \sum_{j=1}^{M} a_j^{(1)} \left\{ \hat{u}_{j,vg}(P) + \int_\Gamma \left(u^*_{,vg} \hat{u}_{j,n} - u^*_{,nvg} \hat{u}_j\right) ds \right\} \tag{24}$$

$$\phi_{y,vg}(P) = -\int_\Gamma \left(u^*_{,vg} \phi_{y,n} - u^*_{,nvg} \phi_y\right) ds + \sum_{j=1}^{M} a_j^{(1)} \left\{ \hat{u}_{j,vg}(P) + \int_\Gamma \left(u^*_{,vg} \hat{u}_{j,n} - u^*_{,nvg} \hat{u}_j\right) ds \right\} \tag{25}$$

Where $v,g = 0,x,y$ and point $p \in \Omega$.

Because equations (18)-(25) involve the boundary quantities w, ϕ_x and ϕ_y, a further manipulation is required to express the boundary stress resultants Q_n, M_n and M_{nt}, and displacements ϕ_n, ϕ_t appearing in equations (7a,b,c) in terms of the deflection w and the rotations ϕ_x, ϕ_y as follows.

$$\phi_n = n_x \phi_x + n_y \phi_y \quad , \tag{26}$$

$$\phi_t = -n_y \phi_x + n_x \phi_y \quad , \tag{27}$$

$$Q_n = (K_s G h)\left(n_x \phi_x + n_y \phi_y + \frac{\partial w}{\partial n}\right) \quad , \tag{28}$$

$$M_n = D\left[n_x\left(\frac{\partial \phi_x}{\partial n} + v\frac{\partial \phi_y}{\partial t}\right) + n_y\left(\frac{\partial \phi_y}{\partial n} - v\frac{\partial \phi_x}{\partial t}\right)\right], \tag{29}$$

$$M_{nt} = D\frac{(1-v)}{2}\left[n_x\left(\frac{\partial \phi_x}{\partial t} + \frac{\partial \phi_y}{\partial n}\right) + n_y\left(\frac{\partial \phi_y}{\partial t} - \frac{\partial \phi_x}{\partial n}\right)\right] \tag{30}$$

where $n_x = cos\theta$ and $n_y = sin\theta$; the angle θ is measured from x to n axis.

Consider the boundary value problem described by equations (6)-(7). There are 8 variables on the boundary that are w, $w_{,n}$, ϕ_x, $\phi_{x,n}$, $\phi_{x,t}$, ϕ_y, $\phi_{y,n}$, $\phi_{y,t}$. We have 5 boundary integral equations (18)-(22)

and 3 equations from the prescribed boundary conditions (7a,b,c). Thus, we can solve this problem and obtain the boundary quantities in terms of the 3M unknown coefficients $a_j^{(l)}$.

4. NUMERICAL EXAMPLES

In order to demonstrate the accuracy, reliability and applicability of the present methodology, a MATLAB code has been developed and used to analyze thick laminated plate problems. The accuracy of the developed technique is evaluated by comparing the obtained results with those available from analytical solutions or FEM solutions. In this paper, the employed radial basis functions f_j are thin plate splines (TPSs), which are defined as

$$f_j(r) = r^2 \ln r, \quad r = \sqrt{(x-x_j)^2 + (y-y_j)^2} \quad j = 1,2,...,M \tag{31}$$

The thin plate spline f_j and the derivatives of $\hat{u}_j$ involved in equations (18)-(25) are listed in the work of Chinnaboon, et.al. [8]. In the following examples, the nondimensionalized maximum deflections and stresses are $\bar{w} = w_0 \frac{E_2 h^3}{q b^4} \times 10^2$ and $\bar{\sigma}_{xx} = \sigma_{xx} \frac{1}{q}$, respectively. In all examples, N = 200 boundary elements and M = 225 domain points are employed.

Example 1 A Simply supported square plate $a \times b$ with thickness h is subjected to a transverse uniform distributed load q. The plate boundary is hard type simply supported and made of an antisymmetric orthotropic material by following properties.

$E_x = kE_y$, $E_y = 3 \times 10^6$, $G_{xy} = G_{xz} = 0.5E_y$,
$G_{yz} = 0.2E_y$, $\nu_{xy} = \nu_{yx} = 0.25$, $K_s = 5/6$, a/h=10
The present results are contained in Table 1 and Table 2 which are compared with the analytical solution obtained in Reddy [2].

Table 1 Central deflections in the laminated plates in example 1.

Theory / Ex/Ey	$\bar{w}$				
	1	10	20	30	40
Present	4.441	1.608	1.048	0.827	0.709
FDST [2]	4.480	1.678	1.089	0.856	0.731
CLPT [2]	4.172	1.412	0.795	0.548	0.415

Table 2 Stresses in the laminated plates in example 1.

Theory / Ex/Ey	$\bar{\sigma}_{xx}$				
	1	10	20	30	40
Present	0.254	0.685	0.759	0.785	0.797
FDST [x]	0.255	0.681	0.755	0.780	0.791
CLPT [x]	0.251	0.693	0.772	0.795	0.801

Example 2 A five-layer cross-ply (0/90/0/90/0) square plate $0<x<a$ and $0<y<a$ with total thickness h is subjected sinusoidal load $f = q\sin(\pi x/a)\sin(\pi y/a)$ .Each lamina is made of an orthotropic material with parameters as follows:
$E_1=25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu = 0.25$
thickness $h_1 = h_3 = h_5 = h/6$ and $h_2 = h_4 = h/4$.
The present results are compared with FEM for Mindlin theory [4] and from 3-dimensional elasticity solution [5], and also AEM [6] as shown in Table 3.

Table 3 Central deflections in the laminated plates in example 2.

Theory / a/h	Deflection , $\bar{w}$		
	4	10	20
Present	16.375	6.144	4.591
3-D [5]	18.668	6.830	4.981
FEM [4]	15.602	6.201	4.786
AEM [6]	15.399	6.145	4.643

As can be seen from the above two examples, it is clear that the present numerical results are in good agreement with those obtained from the analytical solutions and other numerical solutions (FEM and AEM). Thus, the validity of the proposed method are confirmed.

CONCLUSTIONS

In this paper, a Boundary Element Method has been developed for analysis of thick laminated plates. The present method is based on the concept of the analog equation, which the original governing differential equations are substituted by three uncoupled Poisson's equations with fictitious sources under the same boundary conditions. The formulation has no limitations on plate shapes and boundary conditions. The following conclusions can be drawn from this study:

1. As the method is boundary-only, it has all the advantages of the pure BEM, i.e. the discretization and integration are performed only on the boundary. No subdomains or background cells of the problem domain are

required in either the interpolation or integration.

2. The known fundamental solution of the Laplace equation is employed to derive the integral representation of the solution. Thus, the kernels of the boundary integral equations are known and can readily be evaluated.
3. Accurate numerical results for the displacements and the stress resultants are obtained using Thin Plate Splines (TPSs) as radial basis functions. Therefore, no additional parameter is required to obtain the solution.
4. The concept of the analog equation in conjunction with radial-basis-functions approximation of the fictitious sources renders the BEM a versatile computational method for solving difficult engineering problems. It depends only on the order of the differential equation and not specific differential operator which governs the problem under consideration.

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