# A HEURISTIC ALGORITHM FOR THE ECONOMIC PRODUCTION QUANTITY WITH A PIECEWISE LINEAR DEMAND

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**Abstract** - Short product life cycles, that consists of rapid growth, maturity and decline phases combined with different stage, are becoming very common in many industries especially high technology products. Normally, high technology factories would develop a medium- to long-term production plan that typically covers a time horizon about six months. Therefore, the time horizon may be across over two stages. This study presents a heuristic algorithm for an economic production quantity model with a deterministic piecewise linear function in demand called the "piecewise linear model" that is different from previous inventory models focusing on linear trend in demand with a strict assumption that is no inventory to be held at the end of each stage. Moreover, a piecewise linear function can be approximated a nonlinear function. This study does not concentrate on this, but it provides a simple algorithm to easily deal with this problem. Although our approach is a heuristics, our algorithm is better than the sum of two independent single-piece linear models on the total cost's performance. A general procedure for solving these problems is provided and numeric examples for illustrating our proposed model are given as well.

**Index Terms** - Economic production quantity (EPQ), Linear trend in demand, Piecewise linear demand.

## I. INTRODUCTION

The traditional economic production quantity (EPQ) assumes a fixed demand. Actually, the demand may vary with time. For technology innovation, short product life cycles are becoming very common in many industries especially electronic consumer products among recent years. A typical demand curve for these products consists of rapid growth, maturity and decline phases combined with different stage, which an entire life cycle time may be ranging from nine to eighteen months [1]. In today's competitive business environment, the collaborative planning, forecasting and replenishment (CPFR), that aims to enhance supply chain integration by supporting and assisting joint practice between a buyer and vendor, has become essential. Under operations of the CPFR, the vendor would request the buyer to set up a medium- to long-term proposal replenishment schedule for his collaborative forecast and inventory policy. Therefore, the planner of the vendor has to map out the interval and demand of each stage and set up an intermediate-range production plan according to the buyer's forecast. This planning horizon is perhaps across over two stages on product life cycle. For this phenomenon of product life cycle, most previous studies have assumed that the demand pattern is one linear function on production or inventory problems. However, these researches consider only one linear function over the planning horizon that is not capable of handling the above situation with different demand trend and multiple stages. One primitive solution following previous one linear function algorithm is to sum each separated stage with the single-piece linear model; however, it is neither practical nor effective. To acquire a better solution and release a strict assumption that is no inventory to be held at the end of each stage, this study proposes an algorithm for the production policy when the trend of demand is a piecewise linear function.

To consider an increasing demand on the product life cycle or bloom season, Resh [2] was the first author to introduce an inventory model with time-proportional demand. Donaldson [3] proposed an analytic approach for replenishment problem with a linear (increasing) trend in demand. Based on the above contribution, Henery [4] was the first to put forward a recursive procedure for determining the optimal replenishment schedule under the condition of fixed replenishment lots. Hariga's iterative algorithm [5] replied upon replenishment period that specified the proportion of two adjacent periods for both increasing and decreasing demand. Hill [6] was the first to investigate how batching policies were determined under a linearly increasing demand under a finite production rate. Omar and Smith [7] modified Hill's model [6] for linearly increasing demand in order to derive the optimal solution for integrating the batch size of raw materials and production lots in a manufacturing system. Rau and Ouyang [8] presented a general and simple algorithm to obtain an optimal solution for three inventory models on replenishment-batching, production-batching and integrated both replenishment and production batching policy in a manufacturing system.

Actually, a piecewise linear function can described as an approximation of nonlinear function. To break the demand pattern up into segments, Hill [9] introduced

a general, time-varying, continuous, deterministic demand pattern through a complete product life cycle. It then uses a conceptually straightforward approach to determine the stock replenishment schedule. Yang, Zhao and Rand [10] developed an analytic algorithm to deal with a replenishment problem for a non-linear increasing (quadratic equation) demand pattern. Wang [11] provided a consecutive improvement approach to solve the same problem that obtained a better cost's performance. Following their previous work, Yang, Zhao and Rand [12] presented a parametric eclectic approach for a non-linearly decreasing to solve this problem. Rau and Ouyang [13] investigated how to model an economic order quantity with a piecewise linear trend in demand. Therefore, none of the above researches can solve the proposed EPQ problem. In this paper, we extend this algorithm Rau and Ouyang [13] to this piecewise case and provide a simple heuristics to easily solve economic production quantity problem. In subsequent sections, first, we briefly review the classical model with a linear trend in demand. Then, we extend this algorithm to a multi-stage and piecewise linear function model. Finally, we conduct the procedures to derive the solution of these two models and numeric examples are given to illustrate these linear models.

## II. ASSUMPTIONS AND NOTATION

To develop our proposed model, the following assumptions and notation are used.

### A. Assumptions

- A finite foreseeable time horizon is considered.
- The demand is a continuous piecewise function that consists of two pieces, each of which is a constant, linearly increasing or decreasing function.
- The ending point of the first stage is the starting point of the second stage and time is continuous.
- Unit production cost is deterministic and fixed.
- Production capacity is always greater than demand.
- A single item is considered.
- Lead-time or shortage is not considered.
- Demand is always greater than zero at the end of the time horizon.
- No stock is held at the beginning and the end of the time horizon.

### B. Notation

n number of production cycles.
$n^*$ optimal production cycles.
$n_0$ number of production cycles obtained by EOQ model.
$H_1$ the length of the first stage.
H planning horizon.
W total cost, including set up and holding cost.
$W^*(n)$ optimal total cost for n production cycles.
$c_1$ set up cost per order.
$c_2$ holding cost per unit per year.
$f_1(t)$ demand function during the first stage.
$f_2(t)$ demand function during the second stage.
$a_1$ demand at t = 0
$b_1$ rate of demand change per unit of time during stage one.
$b_2$ rate of demand change per unit of time for at stage two.
$t_i$ Terminating time of ith production cycles time or starting time of i+1th production cycles period i = 0, 1,…, n-1.
$T_i$ interval of the ith production cycle, i = 1,2,…,n, where $T_i = t_i - t_{i-1}$.
$T_k$ length of the kth production cycles period across stage one and stage two.
$f_m(t_i)$ demand at midpoint of the ith production cycle and $f_m(t_i) = f(\frac{t_i + t_{i-1}}{2})$.

## III. THE MATHEMATICAL DEVELOPMENT

### A. Single-piece Linear Model

The economic production quantity model for a linear trend in demand is an unconstrained nonlinear problem, where we have to determine the production schedule $t_i$ and number of production cycles n over a fixed planning horizon H. A specified production cycles n is assumed in the following mathematical development. The total relevant cost, including production set up cost and holding cost, is given by [8]:

$$W(n) = nc_1 + c_2 \sum_{i=0}^{n-1} \left[ \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} f(u)dudt - \left( \int_{t_i}^{t_{i+1}} f(t)dt \right)^2 / 2P \right] \quad i = 1, 2, \ldots n-1. \tag{1}$$

First, we solve the best production schedule. Under n fixed differentiate with respect to $t_i$, i =1, 2,….n - 1 and one can obtain:

$$\frac{\partial W(n)}{\partial t_i} = c_2 \left[ (t_i - t_{i-1}) f(t_i) - \int_{t_i}^{t_{i+1}} f(t)dt + f(t_i) \left( \int_{t_i}^{t_{i+1}} f(t)dt - \int_{t_{i-1}}^{t_i} f(t)dt \right) / P \right]. \tag{2}$$

To find the conditions for a stationary value of W, let (2) equal zero and get:

$$(t_i - t_{i-1}) f(t_i) - Q_{i+1} + \frac{f(t_i)(Q_{i+1} - Q_i)}{P} = 0. \tag{3}$$

According to a trapezoidal or triangular area computation, the ith production quantity is:

$$Q_i = (t_i - t_{i-1})\frac{f(t_{i-1}) + f(t_i)}{2} = (t_i - t_{i-1}) f(\frac{t_{i-1} + t_i}{2}) = f_m(t_i)(t_i - t_{i-1}) \quad (4)$$

Substituting (4) into (3), we obtain the following equation:

$$PT_i f(t_i) - Pf_m(t_{i+1})T_{i+1} = f(t_i)\left(f_m(t_{i+1})T_{i+1} - f_m(t_i)T_i\right) \quad (5)$$

and the proportion of $T_{i+1}$ to $T_i$ is as follows:

$$T_{i+1} = \frac{f(t_i)(P - f_m(t_i))}{f_m(t_{i+1})(P - f(t_i))}T_i \quad (6)$$

and $$f(t_i) = f\left(\sum_{k=1}^{i} T_k\right)$$

It is clear that once $T_1$ is known, all remaining production periods can be determined using (6). Based on $T_1$ and the planning horizon H, the optimal production period $T_i$ under the number of production cycles n is:

$$T_{i+1} = \prod_{j=1}^{i} \frac{f(t_j)(P - f_m(t_j))}{f_m(t_{j+1})(P - f(t_j))}T_1 \quad \text{for} \quad f(t_j) = f\left(\sum_{k=1}^{i} T_k\right) \quad \text{and} \quad \sum_{i=1}^{n} T_i = H \quad (7)$$

**B. Piecewise Linear Model**

In light with the collaborative planning, forecasting and replenishment, we suppose that the vendor have a planning time horizon H. The demand function during the first stage is $f_1(t) = a_1 + b_1 t$ with a known period $H_1$ and the demand function during the second stage (from $H_1$ to H) is $f_2(t) = a_2 + b_2 t$ and $a_2 = a_1 + (b_1 - b_2)H_1$ from the buyer's forecast. The ending point of the first stage is the starting point of the second stage. Hence, the demand function under the planning horizon can be expressed as follows:

$$f(t) = \begin{cases} a_1 + b_1 t & t < H_1 \\ a_2 + b_2 t & else \end{cases}. \quad (8)$$

The demand function f(t) is a continuous function of time t but not differentiable only at $H_1$. So, it has to be separately processed by the following two cases.

**Case one:** If the kth production cycle (k is less than n) precisely occurs at $H_1$, this model can be deemed two single-segment linear models. Following the single-segment linear algorithm as shown in Single-piece linear model, the optimal solution is the sum of the total cost W from two segments. We modify (7) and easily acquire the best production schedule as the following equation:

$$T_{i+1} = \begin{cases} \prod_{j=1}^{i} \frac{f_1(t_j)(P - f_{1m}(t_j))}{f_{1m}(t_{j+1})(P - f_1(t_j))}T_1 & \text{and} \quad i = 1, 2, \ldots k-1 \\ \prod_{j=k}^{i} \frac{f_2(t_j)(P - f_{2m}(t_j))}{f_{2m}(t_{j+1})(P - f_2(t_j))}T_k & \text{and} \quad i = k, \ldots, n-1 \end{cases}$$

for $f(t_j) = f\left(\sum_{k=1}^{i} T_k\right)$ and $\sum_{i=1}^{n} T_i = H$

**Case two:** Under a fixed number of production cycles n in the time horizon, suppose the kth production period is across segments one and two and there are two linear demand functions $f_1(t)$ and $f_2(t)$ at the kth production period, as shown in Fig. 1 for both (a) increasing and (b) decreasing demands. The inventory level expression for Figure 1 is shown in Figs. 2 and 3 for increasing and decreasing demand, respectively. Referring to the above single-piece linear model, the total time-weighted stockholding, including $t \le H_1$, $H_1 \le t \le t_k$ and $t_k \le t$ can be expressed

$$\sum_{i=0}^{k-2}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_1(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_1(t)dt\right)^2 / 2P\right]$$

when $t \le H_1$

$$\int_{t_{k-1}}^{H_1}\int_{t}^{H_1} f_1(u)dudt + (H_1 - t_{k-1})\int_{H_1}^{t_k} f_1(t)dt + \int_{H_1}^{t_k}\int_{t}^{t_k} f_2(u)dudt - \left(\int_{t_i}^{H_1} f_1(t)dt + \int_{H_1}^{t_k} f_2(t)dt\right)^2 / 2P$$

when $t_i < H_1$ and $t_{i+1} > H_1$

$$\sum_{i=k}^{n-1}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_2(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_2(t)dt\right)^2 / 2P\right]$$

when $t_k \le t$

(10)

Thus, the total cost, including the set up cost and holding cost, is given by

$$W = nc_1 + c_2\sum_{i=0}^{k-2}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_1(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_1(t)dt\right)^2 / 2P\right] + c_2\left[\int_{t_{k-1}}^{H_1}\int_{t}^{H_1} f_1(u)dudt + (H_1 - t_{k-1})\int_{H_1}^{t_k} f_1(t)dt + \int_{H_1}^{t_k}\int_{t}^{t_k} f_2(u)dudt - \left(\int_{t_i}^{H_1} f_1(t)dt + \int_{H_1}^{t_k} f_2(t)dt\right)^2 / 2P\right] + c_2\sum_{i=k}^{n-1}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_2(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_2(t)dt\right)^2 / 2P\right] \quad (11)$$

Here, our heuristic method conjectures that W is differentiable and it can be found its minimum at $\partial W / \partial t_i = 0$. Differentiate (11) with respect to $t_i$ and we have

$$\frac{\partial W(n)}{\partial t_i} = c_2\left[(t_i - t_{i-1})\, f_1(t_i) - \int_{t_i}^{t_{i+1}} f_1(t)dt + f(t_i)\left(\int_{t_i}^{t_{i+1}} f_1(t)dt - \int_{t_{i-1}}^{t_i} f_1(t)dt\right)/P\right]. \qquad i = 1,..,k-2$$

$$\frac{\partial W(n)}{\partial t_i} = c_2\left[(t_i - t_{i-1})\, f(t_i) - \int_{t_i}^{t_{i+1}} f(t)dt + f(t_i)\left(\int_{t_i}^{t_{i+1}} f(t)dt - \int_{t_{i-1}}^{t_i} f(t)dt\right)/P\right]. \qquad i = k-1$$

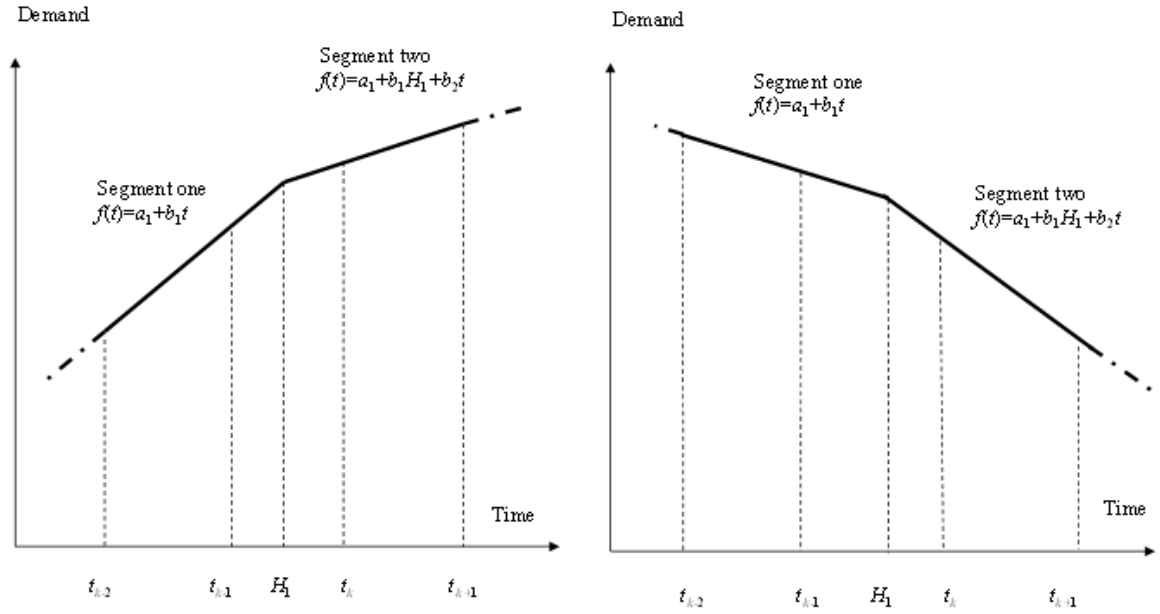
and

$$\frac{\partial W(n)}{\partial t_i} = c_2\left[(t_i - t_{i-1})\, f_2(t_i) - \int_{t_i}^{t_{i+1}} f_2(t)dt + f(t_i)\left(\int_{t_i}^{t_{i+1}} f_2(t)dt - \int_{t_{i-1}}^{t_i} f_2(t)dt\right)/P\right]. \qquad i = k,..,n-1$$

, (12)

We summarize (12) and derive the production schedule in the following equation (13).

## IV. SOLUTION PROCEDURE




**(a) Increasing demand** **(b) Decreasing demand**

**Fig. 1.The kth cycle across Segments one and two in the piecewise linear demand model**

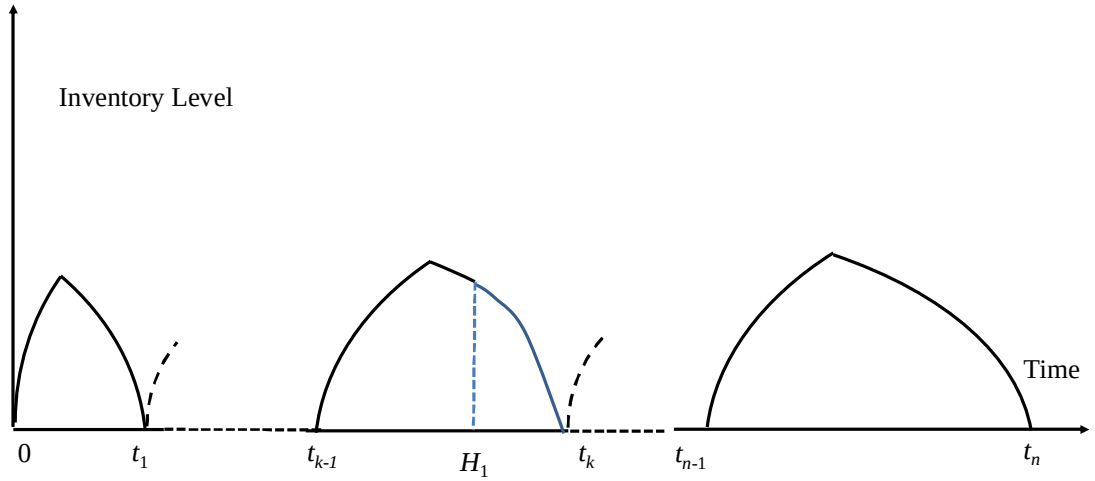



**Fig. 2. The inventory level for increasing demand of the Fig. 1.**

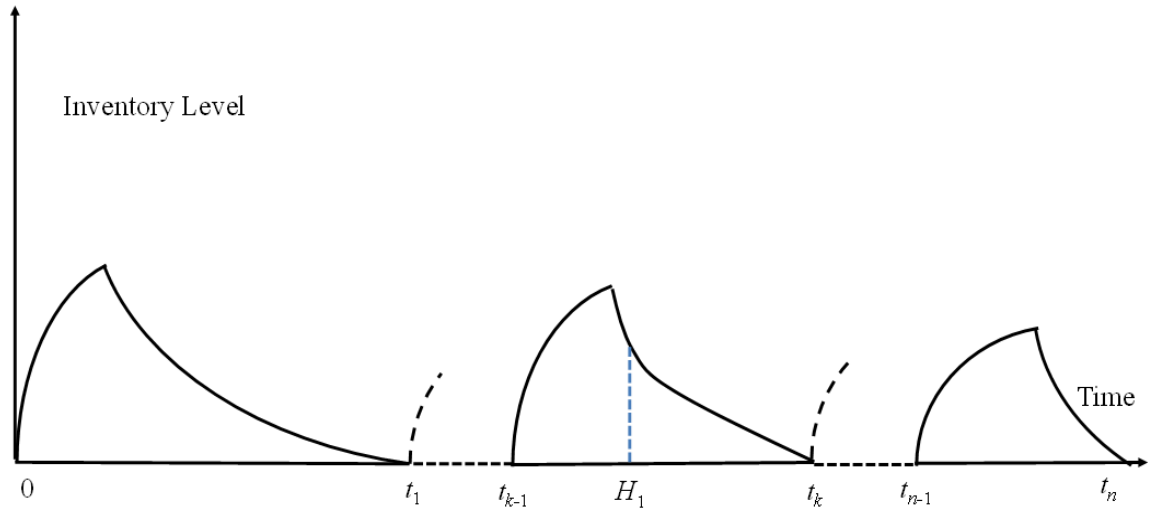



**Fig. 3. The inventory level for decreasing demand of the Fig. 1.**

$$T_{i+1}=\begin{cases}\prod_{j=2}^{i}\dfrac{f_1(t_j)(P-f_{1m}(t_j))}{f_{1m}(t_{j+1})(P-f_1(t_j))}T_1 & t_{i+1}<H_1\\ \dfrac{\left(f_1((\frac{t_{k-1}+H_1}{2})(H_1-t_{k-1})+f_2((\frac{t_k+H_1}{2})(t_k-H_1)\right)}{f_1((t_i)T_{k-1}} & t_i<H_1 \quad and \quad t_{i+1}>H_1\\ \prod_{j=k}^{i}\dfrac{f_2(t_j)(P-f_{2m}(t_j))}{f_{2m}(t_{j+1})(P-f_2(t_j))}T_k & t_{i+1}>H_1\end{cases} \quad (13)$$

for $f(t_j)=f\left(\sum_{k=1}^{i}T_k\right)$ and $\sum_{i=1}^{n}T_i=H$

| Example | P | n | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1000 | 5 | 0.236191 | 0.427126 | 0.606899 | 0.790856 | 1 | | |
| | 2000 | 6 | 0.235224 | 0.415383 | 0.574174 | 0.721671 | 0.862648 | 1 | |
| | ∞ | 7 | 0.230052 | 0.398462 | 0.541279 | 0.669022 | 0.786457 | 0.896232 | 1 |

**Table I shows the production schedules for Single-piece linear example 1**

| Example | P | n | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ |
|---|---|---|---|---|---|---|---|
| 2 | 1000 | 4 | 0.4031 | 0.6456 | 0.8369 | 1 | |
| | 2000 | 3 | 0.4924 | 0.7769 | 1 | | |
| 3 | 1000 | 5 | 0.1464 | 0.3086 | 0.48929 | 0.7121 | 1 |
| | 2000 | 4 | 0.1853 | 0.3974 | 0.6529 | 1 | |

**Table II shows the production schedules for Piecewise linear examples 2 and 3**

Based on the above mathematical model development, the optimal solution of the inventory production policy for the single-piece linear model and the heuristic solution for the piecewise linear model can be conducted in a general procedure as follows:

**Step 1.** Calculate $n_0$ and let $n = n_0$, where $n_0$ is an EOQ-based estimator, as shown in the following equation in order to elevate the searching performance.

$$n_0=\left\lceil\sqrt{\frac{c_2H_1(a_1+b_1H_1)}{2c_1}}\right\rceil+\left\lceil\sqrt{\frac{c_2(H-H_1)[(a_1+b_1H_1)+b_2(H-H_1)]}{2c_1}}\right\rceil \quad (14)$$

**Step 2.** $t_n = H$ and $T_1=\dfrac{H}{n}$.

**Step 3** $T_1=\dfrac{t_n}{n}$.

**Step 4.** Solve $T_i$ with equation (13) by an iterative procedure until a satisfied EPS occurs. Compute the total cost $W^*(n)$ from equation (11).

**Step 5.** Compute the total cost $W^*(n-1)$ and $W^*(n+1)$ by performing Steps 3 and 4 with respect to n-1 and n+1.

**Step 6**. Select Case Min{$W^*(n-1)$, $W^*(n)$, $W^*(n+1)$}

Case $W^*(n-1)$

n = n-1

If n = 1, then Go to Step 7, else Go to Step 2.

Case $W^*(n)$

Go to Step 7.

Case $W^*(n+1)$

n = n+1 : Go to Step 2.

End Select

**Step 7**. The number of production cycle is $n^*$. The production schedule is $T_i$, i = 0,1,…,n-1 and the total cost is $W^*(n)$.

## V. NUMERICAL EXAMPLES

In order to illustrate the proposed model, we follow Hill's [6] examples and provide examples as follows:

Single-piece linear:

Example 1.

$f(t) = 900t$, $H = 1$,

$c_1/c_2 = 4.5$, $P = 1000$, 2000 and $\infty$

piecewise linear examples:

Examples 2 and 3

$f(t) = 200+900t$, $H_1 = 0.6$, $a_2 = 540$ $b_2 = 1200$ $c_1/c_2 = 4.5$, $P = 1000$, 2000

$f(t) = 1200- 900t$, $H_1 = 0.6$, $a_2 = 660$ $b_2 = 1000$ $c_1/c_2 = 4.5$, $P = 1000$, 2000

Table I shows the production schedules for Single-segment model and $W(n)/c_2$ are 43.45, 54.20 and 62.63. Table II shows the production schedules for Piecewise linear examples 2 and 3

## CONCLUSION

This study extended the previous work to provide a heuristic solution for the EPQ model with a two-segment linear function in demand. In terms of the future research, people could focus on other possible ways of solving this problem, more complex inventory policies, or new production and supply chain considerations for related inventory problems with the two-segment piecewise linear trend in demand. Meanwhile, actually the two-segment piecewise linear demand model can be easily extended into multi-segment models. In addition, these multi-segment models can also be approximated as a nonlinear function, such as trigonometric, exponential or cubic equation. However, we believe that the approach of our paper serves as a good starting point for all these future researches.

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