EXPLICITLY SAMPLE-EQUIVALENT DYNAMIC MODELS FOR GAUSSIAN MARKOV, RECIPROCAL, AND CONDITIONALLY MARKOV SEQUENCES

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Abstract - The conditionally Markov (CM) sequence contains different classes including Markov, reciprocal, and so-called CML and CMF (two special classes of CM sequences). Each class has its own forward and backward dynamic models. The evolution of a CM sequence can be described by different models. For example, a Markov sequence can be described by a Markov model, as well as by reciprocal, CML, and CMF models. Also, sometimes a forward model is available, but it is desirable to have a backward model for the same sequence (e.g., in smoothing). Therefore, it is important to study relationships between different dynamic models of a CM sequence. This paper discusses such relationships between models of nonsingular Gaussian (NG) CML, CMF , reciprocal, and Markov sequences. Two models are said to be explicitly sample-equivalent if not only they govern the same sequence, but also a one-one correspondence between their sample paths is made explicitly. A unified approach is presented, such that given a forward/backward CML/CMF /reciprocal/Markov model, any explicitly equivalent model can be obtained. As a special case, a backward Markov model explicitly equivalent to a given forward Markov model can be obtained regardless of the singularity/non singularity of the state transition matrix of the model.

Keywords - Conditionally Markov, reciprocal, Markov, Gaussian sequence, dynamic model, explicitly sampleequivalent.

I. INTRODUCTION

Markov processes have been widely used in many different applications for modeling random phenomena. In some problems more general stochastic processses (e.g., reciprocal processes or CM processes) are required. CM and reciprocal processes have been used in many different applications, including stochastic mechanics, intent inference, image processing, trajectory modeling, and acausal systems [1]–[10]. Dynamic models of CM, reciprocal, and Markov processes play a very important role in the application of these processes. This paper elaborates on the relationship between these models. Gaussian CM processes were introduced in [11]. Reciprocal processes were introduced in [12] related to a problem posed by E. Schrodinger [13]–[14]. Later, reciprocal processes were studied more in [15]–[32]. Dynamic models and characterizations of NG CML and CMF sequences were obtained in [33]. A Gaussian sequence is reciprocal if and only if (iff) it is both CML and CMF [29]. A dynamic model with locally correlated dynamic noise governing the NG reciprocal sequence was presented in [22]. Dynamic models with white dynamic noise governing the NG reciprocal sequence were developed in [29]. Consider stochastic sequences defined over [0;N] = f0; 1; : : : ;Ng. For convenience, let the index be time. A sequence is Markov iff conditioned on the state at any time k, the subsequences before and after k are independent. A sequence is reciprocal iff conditioned on the states at any two times k1 and k2, the subsequences inside and outside the interval [k1; k2] are independent. In other words, "inside" and "outside" are independent given the boundaries. A sequence is CML (CMF) iff conditioned on the state at time N (0), the sequence is Markov over [0;N □ 1] ([1;N]). The subscripts "L" ("F") is used because the conditioning is at the last (first) time of the interval. The Markov sequence is a special class of reciprocal sequences. The reciprocal sequence is a special class of CML/CMF sequences. Thus, evolution of a Markov sequence can be governed by a Markov/reciprocal/CML/CMF model. Similarly, evolution of a reciprocal sequence can be governed by a reciprocal/CML/CMF model. Therefore, a CM sequence can have more than one model. These models are equivalent in the sense that they govern the same sequence (i.e., the sequences governed by the models have the same distribution). Also, sometimes only a forward (backward) model is available when the corresponding backward (forward) one is desired or required. These forward and backward models are also equivalent since they govern the same sequence. In some cases, the above definition of equivalent models is not sufficient because it is only about distributions, not each sample path. It is only equivalent for the set, not element-wise equivalent. The two-filter smoothing approach is an example, where to verify the conditions required for derivation, one needs relationship between sample paths of dynamic noises and boundary values of forward and backward Markov models for the same sample path of the sequence [34]–[36]. Given a model and a sample path of its dynamic noise and boundary values1 corresponding to an arbitrary sample path of the sequence, it is desirable to obtain an equivalent model and a sample path of its dynamic noise and boundary values leading to the same

sample path of the sequence. In other words, it is desirable to find relationship between the dynamic noises and boundary values (of two equivalent models) leading to the same sample path of the governed sequence. Therefore, such models with an explicit relationship between their sample paths are said to be explicitly sample-equivalent. It is important to find relationships between the equivalent models because one model can be more easily applicable than the other in some applications. For example, the reciprocal model of [22] is driven by colored noise and not necessarily easy to apply for trajectory modeling [8]–[9]. But the equivalent recirpocal CML model of [29] is driven by white noise and its application is straightforward. Determination of a NG backward Markov model based on its forward model has been the topic of several papers [37]– [41]. Equivalence of the backward Markov model in [37]–[40] was derived based on equality of the second moments calculated by forward and backward models, which does not deal with specific sample paths. [41] presented an explicitly equivalent backward Markov model only for forward models with nonsingular state transition matrices. In the case of a singular state transition matrix, the derivation of [41] does not provide explicit equivalence (i.e., the relationship between sample paths of the dynamic noises and the boundary values) of forward and backward models. In such a case, [41] only provides parameters of the backward model. Given a NG Markov model, an approach was presented in [22] for the determination of an explicitly equivalent reciprocal model with locally correlated dynamic noise governing the same Markov sequence. The main contributions of this paper are as follows. It discusses relationships between dynamic models governing NG CML, CMF , reciprocal, and Markov sequences. A unified approach is presented, such that within these classes given a model, any explicitly equivalent model can be obtained. As a special case, a backward Markov model explicitly equivalent to a forward Markov model can be obtained. Unlike [41], this approach works for both singular and nonsingular state transition matrices. The explicitly equivalent reciprocal model obtained in [22] can be derived by our approach. Section II reviews definitions and models of CML, CMF , reciprocal, and Markov sequences. Also, definition of explicitly sample-equivalent models is presented.

Section III presents an approach to determining explicitly equivalent models. In Section IV, explicitly sample-equivalent forward and backward Markov models, and explicitly sampleequivalent CML and reciprocal models are obtained. Section V contains a summary and conclusions.

II. DEFINITIONS AND PRELIMINARIES

Throughout the paper we consider sequences defined over [0;N]. The following conventions are used:

$$[i,j] \triangleq \{i, i+1, \ldots, j-1, j\}, i, j \in [0,N], i < j$$
$$[x_k]_i^j \triangleq \{x_k, k \in [i,j]\}, \quad [x_k] \triangleq [x_k]_0^N$$
$$x \triangleq [x_0, \ldots, x_N]'$$

Also, C is the covariance matrix of the whole sequence [xk]. The symbol "n" is used for set subtraction. **Definition 2.1:** Two dynamic models are equivalent if they govern the same sequence (i.e., their sequences have the same distribution).

Definition 2.2: Two dynamic models are explicitly sample equivalent if, given a sample path of the dynamic noise and the boundary values of one model for an arbitrary sample path of the governed sequence, a sample path of the dynamic noise and the boundary values of the other model leading to the same sample path of the governed sequence is given explicitly. In other words, if the relation between sample paths of dynamic noises and boundary values of two equivalent models is provided (so that the two models generate the same sample path), they become explicitly equivalent. Forward and backward Markov, reciprocal, CML, and CMF models of [22], [33], [29] are reviewed first. Let [xk] be a zero-mean NG sequence.

1) Markov Model: [xk] is Markov iff

$$x_k = M_{k,k-1}x_{k-1} + e_k^M, k \in [1,N] \quad (1)$$
$$x_0 = e_0^M \quad (2)$$

where $[e_k^M]$ is a zero-mean white NG sequence with covariances M_k.

We have

$$\mathcal{M}x = e^M, \quad e^M = [e_0^{M\prime}, \ldots, e_N^{M\prime}]'$$

where $\mathcal{M}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -M_{1,0} & I & 0 & \cdots & 0 & 0 \\ 0 & -M_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -M_{N-1,N-2} & I & 0 \\ 0 & 0 & 0 & \cdots & -M_{N,N-1} & I \end{bmatrix} \quad (3)$$

A NG sequence with covariance matrix C is Markov iff C^{-1} is tri-diagonal given by (17) below with $D_0 = \cdots = D_{N-2} = 0$.

2) Backward Markov Model: $[x_k]$ is Markov iff

$$x_k = M_{k,k+1}^B x_{k+1} + e_k^{MB}, k \in [0,N-1] \quad (4)$$
$$x_N = e_N^{MB} \quad (5)$$

where $[e_k^{MB}]$ is a zero-mean white NG sequence with covariances M_k^B.

We have

$$\mathcal{M}^B x = e^{MB}, \quad e^{MB} = [e_0^{MB\prime}, \ldots, e_N^{MB\prime}]'$$

where $\mathcal{M}^B$ is the nonsingular matrix

$$\begin{bmatrix} I & -M_{0,1}^B & 0 & \cdots & 0 & 0 \\ 0 & I & -M_{1,2}^B & 0 & \cdots & 0 \\ 0 & 0 & I & -M_{2,3}^B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & -M_{N-1,N}^B \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \quad (6)$$

3) Reciprocal Model: $[x_k]$ is reciprocal iff

$$R_k^0 x_k - R_k^- x_{k-1} - R_k^+ x_{k+1} = e_k^R, k \in [1, N-1] \quad (7)$$

where $[e_k^R]_0^N$ is a zero-mean NG sequence with $E[e_k^R(e_k^R)'] = R_k^0$, $E[e_k^R(e_{k+1}^R)'] = -R_k^+$, $E[e_k^R(e_j^R)'] = 0, |k-j| > 1$, $R_k^+ = (R_{k+1}^-)'$, and the boundary conditions

$$R_0^0 x_0 - R_0^- x_N - R_0^+ x_1 = e_0^R \quad (8)$$

$$R_N^0 x_N - R_N^- x_{N-1} - R_N^+ x_0 = e_N^R \quad (9)$$

where $E[e_N^R(e_0^R)'] = -R_N^+$ and $R_0^- = (R_N^+)'$, and the parameters of model (7) and boundary conditions (8)–(9) lead to a nonsingular sequence.

We have

$$\mathcal{R}x = e^R$$
$$e^R = [e_0^{R\prime}, \ldots, e_N^{R\prime}]'$$

where $\mathcal{R}$ is the nonsingular matrix

$$\begin{bmatrix} R_0^0 & -R_0^+ & 0 & \cdots & 0 & -R_0^- \\ -R_1^- & R_1^0 & -R_1^+ & 0 & \cdots & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^- & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & -R_{N-1}^+ \\ -R_N^+ & 0 & 0 & \cdots & -R_N^- & R_N^0 \end{bmatrix} \quad (10)$$

A NG sequence with covariance matrix C is reciprocal iff its C^{-1} is cyclic tri-diagonal given by (17) with $D_1 = \cdots = D_{N-2} = 0$.

The reciprocal model (7) (with its boundary conditions) is well-posed if its parameters lead to a nonsingular covariance matrix for the whole sequence $[x_k]$.

We call model (7) the reciprocal model, to distinguish it from reciprocal CM_L/CM_F models, defined below.

4) Forward CM_c Models: $[x_k]$ is CM_c iff

$$x_k = G_{k,k-1}x_{k-1} + G_{k,c}x_c + e_k, k \in [1, N] \setminus \{c\} \quad (11)$$

$$x_c = e_c, \quad x_0 = G_{0,c}x_c + e_0 \text{ (for } c = N) \quad (12)$$

where $[e_k]$ is a zero-mean white NG sequence with covariances G_k.

For $c = 0$,

$$\mathcal{G}^F x = e^F \quad (13)$$
$$e^F = [e_0', \ldots, e_N']'$$

where $\mathcal{G}^F$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -2G_{1,0} & I & 0 & \cdots & 0 & 0 \\ -G_{2,0} & -G_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0} & 0 & \cdots & -G_{N-1,N-2} & I & 0 \\ -G_{N,0} & 0 & 0 & \cdots & -G_{N,N-1} & I \end{bmatrix} \quad (14)$$

For $c = N$,

$$\mathcal{G}^L x = e^L \quad (15)$$
$$e^L = [e_0', \ldots, e_N']'$$

where $\mathcal{G}^L$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,N} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \quad (16)$$

A NG sequence with covariance matrix C is CM_L (CM_F) iff its C^{-1} is CM_L (CM_F), defined as follows.

Definition 2.3: A symmetric positive definite matrix is CM_L if it has form (17) and CM_F if it has form (18):

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & D_1 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & D_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & D_{N-2} \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_0' & D_1' & D_2' & \cdots & D_{N-2}' & B_{N-1}' & A_N \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} A_0 & B_0 & D_2 & \cdots & D_{N-2} & D_{N-1} & D_N \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ D_2' & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{N-2}' & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ D_{N-1}' & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_N' & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \quad (18)$$

Here Ak, Bk, and Dk are matrices in general.

Remark 2.1: [xk] is reciprocal iff it obeys (11)–(12) and

$$G_k^{-1}G_{k,c} = G_{k+1,k}'G_{k+1}^{-1}G_{k+1,c} \quad (19)$$

$\forall k \in [1, N-2]$ for $c = N$, and $\forall k \in [2, N-1]$ for $c = 0$. Moreover, $[x_k]$ is Markov iff we also have, for $c = N$,

$$G_0^{-1}G_{0,N} = G_{1,0}'G_1^{-1}G_{1,N} \quad (20)$$

and for $c = 0$,

$$G_{N,0} = 0 \quad (21)$$

5) Backward CM_c Models: $[x_k]$ is CM_c iff

$$x_k = G_{k,k+1}^B x_{k+1} + G_{k,c}^B x_c + e_k^B, k \in [0, N-1] \setminus \{c\} \quad (22)$$

$$x_c = e_c^B, \quad x_N = G_{N,c}^B x_c + e_c^B \text{ (for } c = 0) \quad (23)$$

where $[e_k^B]$ is a zero-mean white NG sequence with covariances G_k^B.

Remark 2.2: $[x_k]$ is reciprocal iff it obeys (22)–(23) and

$$(G_{k+1}^B)^{-1}G_{k+1,c}^B = (G_{k,k+1}^B)'(G_k^B)^{-1}G_{k,c}^B \quad (24)$$

$\forall k \in [1, N-2]$ for $c = 0$, and $\forall k \in [0, N-3]$ for $c = N$. Moreover, $[x_k]$ is Markov iff we also have, for $c = 0$,

$$G_N^B)^{-1}G_{N,0}^B = (G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,0}^B \quad (25)$$

and for $c = N$,

$$G_{0,N}^B = 0 \quad (26)$$

A forward/backward CM_L/CM_F model of a reciprocal (Markov) sequence is called a reciprocal (Markov) forward/backward CM_L/CM_F model.

III. DETERMINATION OF EXPLICITLY EQUIVALENT DYNAMIC MODELS: A UNIFIED APPROACH

Let $[x_k]$ be a CM sequence governed by a model presented in Section II. We have

$$Tx = \xi, \quad \xi = [\xi_0', \ldots, \xi_N']' \quad (27)$$

where the vector ξ includes the dynamic noise and the boundary values and $P = \text{Cov}(\xi)$. Matrix T is determined by parameters of the model. Note that T and P for a model have a specific form. So, parameters of equivalent models (in terms of each other) can be determined easily (see the first step of Proposition 3.1 below). T is nonsingular for the forward and the backward CM_L, CM_F, and Markov models. Also, since the sequence is assumed to be nonsingular, T is nonsingular for the reciprocal model, too [22].

By Definition 2.2, explicit equivalence is mutual, i.e., if model 2 is explicitly equivalent to model 1, then model 1 is also explicitly equivalent to model 2. We have the following proposition for the construction of explicitly equivalent models.

Proposition 3.1: Let a forward, backward CM_L, CM_F, reciprocal, or Markov model be given by

$$T_1 x = \xi \tag{28}$$

where $\xi = [\xi_0', \ldots, \xi_N']'$ includes the dynamic noise and the boundary values, and the covariance of ξ is P_1. Any explicitly equivalent forward or backward CM_L/CM_F/reciprocal/Markov model, denoted by

$$T_2 x = \zeta \tag{29}$$

with $\zeta = [\zeta_0', \ldots, \zeta_N']'$ containing the corresponding dynamic noise and boundary values with covariance P_2, is constructed in two steps:

(1) The parameters of the equivalent model are determined based on[2]

$$T_2' P_2^{-1} T_2 = T_1' P_1^{-1} T_1 \tag{30}$$

Therefore, T_2 and P_2 can be obtained given their forms.

(2) The relationship between the dynamic noises and the boundary values of the two models are obtained through

$$T_2'(P_2)^{-1}\zeta = T_1'(P_1)^{-1}\xi \tag{31}$$

Proof: Step (1): The inverse of the covariance matrix (C^{-1}) of the sequence governed by model (28) is calculated as $E[(T_1x)(T_1x)'] = E[\xi\xi']$, which leads to

$$C^{-1} = T_1'(P_1)^{-1}T_1 \tag{32}$$

The inverse of the covariance matrix of the sequence governed by (29) can be calculated as

$$C^{-1} = T_2'(P_2)^{-1}T_2 \tag{33}$$

In order for the two models to be explicitly equivalent, their governed sequences must have the same covariance matrix; thus we have (30) (i.e., the two models are equivalent). Due to the special structures of T1, P1, T2, and P2, parameters of model 2 can be easily obtained in terms of parameters of model 1 using (30) ((38)–(42) below show such calculations for equivalent Markov models). Then, P2 and T2 are known. Note that parameters of model 2 calculated by (30) (in terms of those of model 1) are unique. It can be easily verified based on (30) for all models. This uniqueness can be also concluded from the definition of conditional expectation. Now, given P2 and T2, the relation between _ and _ should be determined so that the two models are explicitly equivalent.

Step (2): Let P2 and T2 be given. We show how (31) leads to an explicitly equivalent model. First, we show that obtained by (31) has the desired dynamic noise and boundary values, i.e., its covariance is P2. By (31), we have

$$\begin{aligned} &T_2'(P_2)^{-1}\text{Cov}(\zeta)(P_2)^{-1}T_2 \\ &\quad = T_1'(P_1)^{-1}\text{Cov}(\xi)(P_1)^{-1}T_1 \end{aligned}$$

Then, substituting $\text{Cov}(\xi) = P_1$, we obtain

$$\begin{aligned} \text{Cov}(\zeta) &= P_2(T_2')^{-1}T_1'(P_1)^{-1}P_1(P_1)^{-1}T_1(T_2)^{-1}P_2 \\ &= P_2(T_2')^{-1}T_1'(P_1)^{-1}T_1(T_2)^{-1}P_2 \end{aligned}$$

Since $T_1'P_1^{-1}T_1 = T_2'P_2^{-1}T_2$, we have $\text{Cov}(\zeta) = P_2(T_2')^{-1}T_2'(P_2)^{-1}T_2(T_2)^{-1}P_2 = P_2$, which means that ζ has the desired dynamic noise and boundary values.

Second, we show that assuming (31) holds, two models (28) and (29) generate the same sample path of the sequence. Substituting (28) into (31), we obtain $T_1'(P_1)^{-1}T_1x = T_2'(P_2)^{-1}\zeta$. Then, $P_2(T_2')^{-1}(T_1'(P_1)^{-1}T_1)x = \zeta$. Premultiplying both sides by $(T_2)^{-1}$, we have

$$(T_2'(P_2)^{-1}T_2)^{-1}(T_1'(P_1)^{-1}T_1)x = (T_2)^{-1}\zeta \tag{34}$$

Since $T_1'P_1^{-1}T_1 = T_2'P_2^{-1}T_2$, we get $T_2x = \zeta$. Therefore, (29) and (28) are explicitly equivalent if (31) holds. ■

The first step of Proposition 3.1, (30), is to determine an equivalent model, and the second step, (31), makes the model explicitly equivalent. By **Proposition 3.1**, given a model, one can construct an explicitly equivalent model. Assume that two equivalent models generate the same sample path of the governed sequence. What is the relation between sample paths of their dynamic noises and boundary values? The next proposition answers this question.

Proposition 3.2: Let two equivalent forward, backward CML, CMF , reciprocal, or Markov models be given as

$$T_1 x = \xi \tag{35}$$

$$T_2 x = \zeta \tag{36}$$

where $\xi = [\xi_0', \ldots, \xi_N']'$ and $\zeta = [\zeta_0', \ldots, \zeta_N']'$ contain their dynamic noises and boundary values, with covariances P_1 and P_2, respectively, and the nonsingular matrices T_1 and T_2 are determined by the parameters of the corresponding models. If the two models generate the same sample path of the governed sequence, the relationship between sample paths of their dynamic noises and boundary values is as (31).

Remark 3.3: By (30), (31) is equivalent to

$$T_1^{-1}\xi = T_2^{-1}\zeta \tag{37}$$

Although (37) looks simpler, for the construction of explicitly equivalent models, (31) is preferred. This is explained as follows. It can be seen that the matrices P_i, $i = 1, 2$, in (31) corresponding to the forward or backward CM_L, CM_F, and Markov models are block-diagonal, and their inverses are easily calculated. Also, for the reciprocal model, no calculation is needed because we have $P = T$ [22]. However, calculation of the inverse of T_i, $i = 1, 2$, in (37) is not straightforward in general.

Note that Proposition 3.1 and Proposition 3.2 are not restricted to (forward/backward) CM_L, CM_F, reciprocal, and Markov models. The results work for other models satisfying the required conditions.

IV. EXAMPLES OF EXPLICITLY SAMPLE-EQUIVALENT MODELS

For illustration, two examples of explicitly sample equivalent models obtained using the approach of Proposition 3.1 are presented.

A. Forward and Backward Markov Models

Given a forward Markov model (1)–(2) for $[x_k]$, by (30) parameters of a backward Markov model (4)–(5) for $[x_k]$ are obtained in terms of those of the forward one as follows. For
k = 2; 3; : : : ;N,

$$(M_0^B)^{-1} = M_0^{-1} + M_{1,0}' M_1^{-1} M_{1,0} \quad (38)$$

$$M_{0,1}^B = M_0^B M_{1,0}' M_1^{-1} \quad (39)$$

$$(M_{k-1}^B)^{-1} = M_{k-1}^{-1} + M_{k,k-1}' M_k^{-1} M_{k,k-1} - (M_{k-2,k-1}^B)' (M_{k-2}^B)^{-1} M_{k-2,k-1}^B \quad (40)$$

$$M_{k-1,k}^B = M_{k-1}^B M_{k,k-1}' M_k^{-1} \quad (41)$$

$$(M_N^B)^{-1} = M_N^{-1} - (M_{N-1,N}^B)' (M_{N-1}^B)^{-1} M_{N-1,N}^B \quad (42)$$

Then, by (31), the relationship between dynamic noises and boundary values of the two models is:

$$(M_0^B)^{-1} e_0^{MB} = M_0^{-1} e_0^M - M_{1,0}' M_1^{-1} e_1^M \quad (43)$$

$$(M_k^B)^{-1} e_k^{MB} = (M_{k-1,k}^B)' (M_{k-1}^B)^{-1} e_{k-1}^{MB} + M_k^{-1} e_k^M - M_{k+1,k}' M_{k+1}^{-1} e_{k+1}^M, k \in [1, N-1] \quad (44)$$

$$(M_N^B)^{-1} e_N^{MB} = (M_{N-1,N}^B)' (M_{N-1}^B)^{-1} e_{N-1}^{MB} + M_N^{-1} e_N^M \quad (45)$$

By the above equations, given a backward model, one can also obtain the explicitly equivalent forward model. (38)–(42) and (43)–(45) give explicitly equivalent forward and backward models no matter if the state transition matrix is singular or nonsingular. Based on (43)–(45), we can verify the required condition in the derivation of the two filter smoother [34]–[36], for singular and nonsingular state transition matrices.

B. Reciprocal CML and Reciprocal Models

Let a CML model governing a reciprocal sequence $[x_k]$ be given. Taking the first step of Proposition 3.1, parameters of the reciprocal model governing $[x_k]$ are obtained from parameters of the CML model as follows.

$$R_0^0 = G_0^{-1} + G_{1,0}' G_1^{-1} G_{1,0} \quad (46)$$

$$R_k^0 = G_k^{-1} + G_{k+1,k}' G_{k+1}^{-1} G_{k+1,k}, k \in [1, N-2] \quad (47)$$

$$R_{N-1}^0 = G_{N-1}^{-1} \quad (48)$$

$$R_N^0 = G_N^{-1} + \sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} G_{k,N} + G_{0,N}' G_0^{-1} G_{0,N} \quad (49)$$

$$R_k^+ = G_{k+1,k}' G_{k+1}^{-1}, k \in [0, N-2] \quad (50)$$

$$R_{N-1}^+ = G_{N-1}^{-1} G_{N-1,N} \quad (51)$$

$$R_0^- = G_0^{-1} G_{0,N} - G_{1,0}' G_1^{-1} G_{1,N} \quad (52)$$

Then, taking the second step of Proposition 3.1, the relationship between dynamic noises and boundary values of the two models is as follows. For (11)–(12), we have

$$e_0^R = G_0^{-1} e_0 - G_{1,0}' G_1^{-1} e_1 \quad (53)$$

$$e_k^R = G_k^{-1} e_k - G_{k+1,k}' G_{k+1}^{-1} e_{k+1}, k \in [1, N-2] \quad (54)$$

$$e_{N-1}^R = G_{N-1}^{-1} e_{N-1} \quad (55)$$

$$e_N^R = -\sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} e_k + G_N^{-1} e_N - G_{0,N}' G_0^{-1} e_0 \quad (56)$$

By the above equations, given a reciprocal model, one can obtain the explicitly equivalent reciprocal CML model. This is important because the reciprocal CML model can be more easily applicable than the reciprocal model.

SUMMARY AND CONCLUSIONS

Relationships between dynamic models governing different classes of Gaussian conditionally Markov (CM) sequences (including Markov, reciprocal, and so-called CML and CMF sequences) have been studied. One CM sequence can obey different models. Given one, it is desirable to obtain other models for the same sequence. Two models are called equivalent if they govern the same random sequence (i.e., their sequences have the same distribution). In some problems it is not sufficient to have only equivalent models— we need to know the relationship between dynamic noises and boundary values for the equivalent models to have the same sample path of the governed sequence. Two equivalent models are explicitly sample-equivalent if such a relationship is given. A unified approach has been presented, such that given a forward, backward CML, CMF , reciprocal, or Markov model, any explicitly equivalent such model can be obtained. This approach does not require any assumptions (e.g., nonsingularity) about the matrix coefficients of the models. So, unlike [41] (which is restricted to nonsingular state transition matrices), the proposed approach can be used to obtain a backward Markov model explicitly equivalent to a forward Markov model with either a singular or nonsingular state transition matrix.

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