N THE METRIZABILITY OF FUZZY CONE METRIC SPACES

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Abstract- In this study, we prove that for a topological spaces, metrizability, cone metrizability, fuzzy metrizability and fuzzy cone metrizability are equivalent..

Keywords- Metrizability, fuzzy cone metric, complete

I. INTRODUCTION

After Zadeh [6] introduced the theory of fuzzy sets, many authors have introduced and studied several notions of metric fuzziness ([2],[5],[8],[10],[11]) and metric cone fuzziness ([14],[15]) from different points of view.

By modifying the fuzzy metric spaces in sense of George and Veeramani [2], Öner et al. [15] studied the fuzzy cone metric spaces. They showed that fuzzy cone metric spaces generate a topology which is Hausdorff and first countable.

Here we study some properties of this spaces which fuzzy metric version can be found in [13]. After proving a topological space is metrizable if and only if it admits a compatible fuzzy cone metric, we deduce that for a topological spaces, metrizability, cone metrizability, fuzzy metrizability and fuzzy cone metrizability are equivalent. Later we show that a topological space is completely metrizable if and only if it admits a compatible complete fuzzy cone metric.

II. PRELIMINARIES

Let E be a real Banach space, θ be the zero of E and P be a subset of E. Then P is called a cone [7] if and only if

1) P is closed, nonempty and $P \neq \{\theta\}$,
2) If $a,b\in\mathbb{R}$, $a,b\geq 0$ and $x,y\in P$, then $ax + by\in P$,
3) If both $x\in P$ and $x\in P$ then $x=\theta$.

Given cone P, a partial ordering $\preccurlyeq$ on E with respect to P is defined by $x\preccurlyeq y$ if only if $y-x\in P$. $x\prec y$ will stand for $x\preccurlyeq y$ and $x\neq y$, while $x\ll y$ will stand for $y-x\in int(P)$. Throughout this paper, we assume that all cones has nonempty interior.

There are two kinds of cones: normal and nonnormal cones. A cone P is called normal if there exists a constant $K\geq 1$ such that for all $t,s\in E$, $\theta\preccurlyeq t\preccurlyeq s$ implies $\|t\|\leq K\|s\|$ and the least positive number K satisfying this properties is called normal constant of P [7]. It is clear that $K\geq 1$ [12].

According to [4] a binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions;

1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a*1=a$ for all $a\in[0,1]$,
4) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$, $a,b,c,d\in[0,1]$.

According to [2] a fuzzy metric space is a 3-tuple $(X,M,*)$ such that X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions;

FM1) $M(x,y,t)>0$,
FM2) $M(x,y,t)=1$ if and only if $x=y$,
FM3) $M(x,y,t)=M(y,x,t)$,
FM4) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,
FM5) $M(x,y,.):(0,\infty)\to[0,1]$ is continuous,
$x,y,z\in X$ and $t,s>0$.

In [15], we generalized the concept of fuzzy metric space of George and Veeramani by replacing the $(0,\infty)$ interval by int(P) where P is a cone as follows:

A fuzzy cone metric space is a 3-tuple $(X,M,*)$ such that P is a cone of E, X is nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times int(P)$ satisfying the following conditions, for all $x,y,z\in X$ and $t,s\in int(P)$ (that is $t\gg\theta$, $s\gg\theta$)

FCM1) $M(x,y,t)>0$,
FCM2) $M(x,y,t)=1$ if and only if $x=y$,
FCM3) $M(x,y,t)=M(y,x,t)$,
FCM4) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$,
FCM5) $M(x,y,.):int(P)\to[0,1]$ is continuous.

If $(X,M,*)$ is a fuzzy cone metric space, we will say that M is a fuzzy cone metric on X.

In [15], it was proven that every fuzzy cone metric spaces $(X,M,*)$ induces a Hausdorff first countable topology τ_M on X which has as a base the family of sets of the form $\{B(x,r,t):x\in X,0<r<1,t\gg\theta\}$, where $B(x,r,t)=\{y\in X:M(x,y,t)>1-r\}$ for every r with $0<r<1$ and $t\gg\theta$.

A fuzzy cone metric space $(X,M,*)$ is called complete if every Cauchy sequence is convergent, where a sequence $\{x_n\}$ is said to be a Cauchy if for any $0<\varepsilon<1$ and any $t\gg\theta$ there exists a natural number n_0 such that $M(x_n,x_m,t)>1-\varepsilon$ for all $n,m\geq n_0$ and is said to converge to x if for any $t\gg\theta$ and any $r\in(0,1)$ there exists a natural number n_0 such that $M(x_n,x,t)>1-r$ for all $n\geq n_0$ [15].

According to [7] a cone metric space is an ordered (X,d), where X is any set and $d:X\times X\to E$ is a mapping satisfying:

CM1) $\theta\preccurlyeq d(x,y)$ for all $x,y\in X$,
CM2) $d(x,y)=\theta$ if and only if $x=y$,
CM3) $d(x,y)=d(y,x)$ for all $x,y\in X$,
CM4) $d(x,z)\preccurlyeq d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Let (X,d) be a cone metric space with normal constant 1. Define $a*b=ab$ for every $a,b\in[0,1]$, and let M_d be the function defined on $X\times X\times int(P)$ by

$$M_d(x,y,t)=((\|t\|)/(\|t\|+\|d(x,y)\|))$$

Then $(X,M_d,*)$ is a fuzzy cone metric space, and M_d is called the standard fuzzy cone metric induced by d [16].

Corollary 1 ([16]) Let (X,d) be a cone metric space with normal constant 1and $(X,M_d,*)$ be the standard fuzzy cone metric space induced by d. Then (X,d) is complete if and only if $(X,M_d,*)$ is complete.

Lemma 2 ([1]) A T_1 topological space (X,t) is metrizable if and only if it admits a compatible uniformity with a countable base.

Lemma 3 ([15]) Let (X,M,*) be a fuzzy cone metric space. Then t_M is a Hausdorff topology and for each $x\in X$, $\{B(x,1/n,t/n) : n\in N \text{ and } t\gg\theta \text{ constant}\}$ is a neighborhood base at x for the topology t_M.

III. RESULTS

Theorem 4 Let (X,M,*) be a fuzzy cone metric space. Then (X,M,*) is a metrizable topological space.

Proof. Let $t\gg\theta$ constant. For each $n\in N$ define
$U_n = \{(x,y)\in XxX : M(x,y,t/n) > 1-1/n\}$
We need to show that $\{U_n : n\in N\}$ is a base for a uniformity U on X whose induced topology coincides with t_M. For each $n\in N$, notice that following is obvious ,
$\{(x,x) : x\in X\}\grave{I}U_n$
$U_{n+1}\grave{I}U_n$
$U_n = U_n^{-1}$

Since * is continuous, there is an $m\in N$ such that $m > 2n$ and
$(1-(1/m))*(1-(1/m))>1-(1/n)$.
Let $(x,y)\in U_m$ and $(y,z)\in U_m$. Since M(x,y,.) is nondecreasing [15], $M(x,z,t/n)>M(x,z,2t/m)$ and we have

$$\begin{aligned} M(x,z,t/n) &\geq M(x,y,t/m)*M(y,z,t/m) \\ &\geq (1-(1/m))*(1-(1/m)) \\ &> 1-(1/n) \end{aligned}$$

Therefore $(x,z)\in U_n$ and $U_m\circ U_m\grave{I}U_n$. Hence $\{U_n : n\in N\}$ is a base for a uniformity U on X. For each $x\in X$ and each $n\in N$, we have

$$\begin{aligned} U_n(y) &= \{x\in X : M(x,y,t/n)>1-1/n\} \\ &= \{B(x,1/n,t/n) : n\in N \text{ and } t\gg\theta \text{ constant}\}. \end{aligned}$$

By Lemma 3, the topology induced by U coincides with t_M and by Lemma 2, (X,M,*) is a metrizable topological space.

Corollary 5 A topological space is metrizable if and only if it admits a compatible fuzzy cone metric.

Proof. Let (X,t) is a metrizable space and d be a metric on X compatible with t. Then, (X,d) is a cone metric space with the normal constant 1 [7] and the standard fuzzy cone metric space $(X,M_d,*)$ induced by d is compatible with t ([16]). The converse is obvious by Theorem 4.

Corollary 6 Let (X,t) be a topological space. Then followings are equavalent:
i (X,t) is a metrizable space.
ii (X,t) is a cone metrizable space.
iii (X,t) is a fuzzy metrizable space.
iv (X,t) is a fuzzy cone metrizable space.

Proof. By the Theorem 3.4 in [9] iÛii. By the Corollary 1 in [13], iÛiii. By the Corollary 5, iÛiv.

Corollary 7 A topological space is completely metrizable if and only if it admits a compatible complete fuzzy cone metric.

Proof. Let (X,M,*) be a complete fuzzy cone metric space. In the proof of Theorem 4, we proved that $\{U_n : n\in N\}$ is a base for a uniformity U on X compatible with t_M, where for every $n\in N$, $U_n=\{(x,y)\in XxX : M(x,y,t/n)>1-1/n;\ t\gg\theta \text{ constant}\}$. Therefore there exists a metric d on X whose induced uniformity coincides with U. Now we need to prove that d is complete. Let $\{x_n\}$ be a Cauchy sequence in (X,d). For fixed r,t, with $0<r<1$ and $t\gg\theta$ we choose a $k\in N$ such that $1/k<r$ and $t/k\|t\|<t$. Then, there is $n_0\in N$ such that $(x_n,x_m)\in U_k$ for every $n,m>n_0$. For each $n,m>n_0$, we have
$M(x_n,x_m,t) \geq M(x_n,x_m,1/k)>1-(1/k)>1-r$.
Hence $\{x_n\}$ is a Cauchy sequence in (X,M,*). Since (X,M,*) is complete, it is convergent with respect to t_M. Therefore, d is a complete metric on X. We deduced that (X,t_M) is completely metrizable. On the other hand, let (X,t) be a completely metrizable space and d be a complete metric on X compatible with t. Since every metric space is a cone metric spaces with normal constant 1, by Corollary 1, the standard fuzzy cone metric (X,M,*) induced by d is complete, and it is compatible with t.

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