SPEKTRUMOF P-HYPONORMAL OPERATOR

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Abstract - Given all bounded linear operators on Hilbert space denoted B(H) and $T \in B(H)$ .This study aimed to investigate the spectrum of p- hypo normaloperatorsT and their properties in Hilbert space. The spectrum set in question includes approximate spectrum, point spectrum, residual spectrum, and joint point spectrum.

I. INTRODUCTION

Given a Hilbert space X and Y over the same field, namely F. Field F which is in this paper is ℝ/C. The set of all bounded linear operators from X to Y is written by $B(X,Y)$. Further, in the event that X = Y, $B(X,X)$ is written by $B(X)$ or $B(Y)$.

Given a Hilbert space H over the field Fand $T \in B(H)$, for some p > 0 bounded linear operator T, which has property of $(T^*T)^p \geq (TT^*)^p$ is named as p- hyponormal operator. If p = 1, then T is hyponormal operator[1],[2], [3],[4]. The resolven set of T, written as $\rho(T)$, is set of all regular values λ of T. Complement of $\rho(T)$, written as $\sigma(T) = C \setminus \rho(T)$, is called spectrum of T and $\lambda \in \sigma(T)$ is called spectral value of T. Further, spectrum of $\sigma(T)$ is grouped into several sets which is odd each other i.e. point spectrum or discrete spectrum is named as $\sigma_r(T)$, approximate spectrum is named as $\sigma_{ap}(T)$, residual spectrum is named as $\sigma_r(T)$, and joint point spectrum is named as $\sigma_{jp}(T)$.

In [5], point spectrum or discrete spectrum is sets of all λ, so $R_\lambda(T)$ is not exist. The values of $\lambda \in \sigma_p(T)$ are eigen values of T with $R_\lambda(T) = (T - \lambda I)^{-1}$ and I is an identity operator on Hilbert space. The approximate spectrum is set of all values $\lambda \in C$, so there is a sequence (x_n) in H that fulfills characteristic of $\|x_n\| = 1$, for every $n \in N$ and $\lim_{n\to\infty} \|T(x_n) - \lambda(x_n)\| = 0$. The residual spectrum is set of all $\lambda \in C$, so $T_\lambda = T - \lambda I$ is injective but $\overline{R(T_\lambda)} \neq H$. Meanwhile, the joint point spectrum is the set of all $\lambda \in C$, so there is $x \in H$ which is not zero that fulfills $T(x) = \lambda x$ and $T^*(x) = \bar{\lambda} x$. Based on the description, it will be investigated characteristic of p-hyponormal operator spectrum in Hilbert spaces for p> 0.

II. SPEKTRUM OPERATOR HYPO NORMAL-P

This following, it will be discussed examples and definitions about point spectrum of p- hyponormal operator and its characteristic in Hilbert space. Firstly, it will be discussed about definition of proper vector, proper value, and value-AP [6]. Earlier, it will be delivered about definition of hypo normal operator and p-hypo normal operator, also its relationship in Hilbert spaces[1],[2], [3], [4].

Definition 1[1],[4]. Givenspace Hilbert H and $T \in B(H)$. T operator ishyponormal operator,if $T^*T \geq TT^*$. T is hypo normal operator equivalently, if $|T|^2 \geq |T^*|^2$ with $|T|^2 = T^*T$ and T^* isadjoinofToperator.T operator can be decomposed into $T = U|T|$ withUis operator unitary. If $T = U|T|$, then $N(U) = N(|T|)$.

Definition2[2]. Givenspace Hilbert H and $T \in B(H)$. Forp > 0, T is called p- hypo normal operator if $(T^*T)^p \geq (TT^*)^p$. T is p- hypo normal operator equivalently, if $|T|^{2p} \geq |T^*|^{2p}$ with $|T|^2 = T^*T$ and T^* is adjoinofT operator.it is clear that $|T|^2$ is positive operator [7].

Definition3.Givenspace Hilbert H and $T \in B(H)$.

1. Vector of $x \in H$ is called"proper vector"Toperator,if $x \neq \theta$ andthere are scalar of λ, so $T(x) = \lambda x$.

2. Scalar of λ is called "proper value" Toperator,ifthere are vector of $x \neq \theta$, so $T(x) = \lambda x$.

FromDefinition1, ifx is proper vector,then there is λ scalar so $T(x) = \lambda x$. Considering that γ scalar fulfills $T(x) = \gamma x$. As a result,

$$\lambda x = \gamma x \Leftrightarrow (\lambda - \gamma) x = \theta .$$

Because $x \neq \theta$ then $\lambda - \gamma = 0$. Thus, $\lambda = \gamma$ meanseveryx is proper vector mentioningwithsingle λ scalar, so $T(x) = \lambda x$.

Further, Givenspace Hilbert H and $T \in B(H)$, also λ is proper value of T operator. It means that there is $x \in H, x \neq \theta$, so $T(x) = \lambda x$. Because of $x \neq \theta$, then $\left\| \frac{x}{\|x\|} \right\| = 1$ and

$T\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|} T(x) = \frac{1}{\|x\|} \lambda x = \lambda \frac{x}{\|x\|}$. Thus, it can be concluded that:

λ isproper value of Toperator $\Leftrightarrow$ there is $x \in H$ with $\|x\| = 1$ so $T(x) = \lambda x \left(\|T(x) - \lambda x\| = 0 \right)$.

Based on that explanation, later it will be formeddefinitionabout setof $\sigma_{ap}(T)$.

Definition4[5]. Givenspace Hilbert H and $T \in B(H)$. The approximate spectrum set fromT ,written as $\sigma_{ap}(T)$, is set of all values $\lambda \in C$, so there is a sequence of (x_n) on H which fulfills this following characteristics:

1) $\|x_n\| = 1$, forevery $n \in N$,

2) $\lim_{n \to \infty} \|T(x_n) - \lambda (x_n)\| = 0.$

Furthermore, λ scalar is usuallycalled value-AP of Toperator.

In this example 3, it is understood that classic Hilbert space isseparabelHilbert space. Later, the separabelHilbert space is Hilbert space which has total sequence. Because Hilbert spaceH is sparabel, thenH hasorthonormal sequence.

Example5.Givenclassic Hilbert space H, means that there is $\{x_n\}$ which is basis orthonormal H. Taken $\lambda \in C$ anddefined operator $T : H \to H$ with $T(x_n) = \mu_n x_n$ and $\mu_n \to \lambda$. Obtained:

1) $\|x_n\| = 1$, forevery $n \in N$,

2) $\|T(x_n) - \lambda(x_n)\| = \|\mu_n x_n - \lambda x_n\| = |\mu_n - \lambda| \|x_n\| = |\mu_n - \lambda| \to 0.$

So, $\lambda \in C$ is approximate spectrum of T.

Afterwards, it will begivendefinitionabout compact operator.

Definition6.Known thatHilbert space Hand $T \in B(H)$. T operator is compactif $(T(x_n))$ has part of sequence which is convergent, for every bounded sequence (x_n) on H.

This following, it will be discussed spectrum characteristics ofbounded linear operator onHilbert space.

Theorem7.If $T \in B(H)$ is compact operator and $\lambda \in \sigma_{ap}(T) \setminus \{0\}$, then $\lambda \in \sigma_p(T) \setminus \{0\}$.

Proof: Known $\lambda \in \sigma_{ap}(T) \setminus \{0\}$, means that there is a sequence of (x_n) then $\|x_n\| = 1$ and $\|T(x_n) - \lambda x_n\| \to 0$. BecauseT is compact,thensequence of $(T(x_n))$ haspart of sequence which is convergent, saythat sequence of $(T(x_{n_k})) \subseteq (T(x_n))$ is convergent into u. Obtained,

$$\|u - \lambda(x_{n_k})\| \leq \|u - T(x_{n_k})\| + \|T(x_{n_k}) - \lambda(x_{n_k})\| \to 0.$$

So, $\lambda(x_{n_k}) \to u.$

Because T is continuous then $T(\lambda x_{n_k}) \to T(u)$. So,

$T(u) = T\left(\lim_{k \to \infty} \lambda x_{n_k}\right) = \lambda \lim_{k \to \infty} T(x_{n_k}) = \lambda u$ and

$\|u\| = \left\|\lim_{k \to \infty} \lambda x_{n_k}\right\| = |\lambda| \lim_{k \to \infty} \|x_{n_k}\| = |\lambda| > 0.$ □

Afterwards, it will be given about space concept of invariant, restriction, and reduction. The concept will be used for proving spectrum characteristics of bounded linear operator on a Hilbert space[8].

Definition 8.Given Hilbert spaceH, $T \in B(H)$, and $N \subseteq H$ close sub space. Sub spaceof N is said that it is invariant towardsT operator if

$$T(N) \subseteq N .$$

Definition 9.Given a Hilbert spaceH, $T \in B(H)$, $N \subseteq H$ the closed section of the room, andN invariant against T. Operator $T|_N : N \to N$ defined $T|_N(x) = T(x), \forall x \in N$ called a restriction T on N.

Definition 10.Given a Hilbert space H, $T \in B(H)$, and $N \subseteq H$ the closed section of the room. The closed section of the room N it said the reduction T when N and $N^{\perp}$ invariant against T . Example 11. Given a Hilbert space H, $T \in B(H)$, andT normal. Given

$N = \overline{T(H)} = \overline{T^*(H)} = \{x \in H : T(x) = \theta\}^{\perp}$ the closed section of the room H . It will be shown Nthe reduction of T This means that the N and $N^{\perp}$ invariant against T .

1. Taken any $y \in \{x \in H : T(x) = \theta\}$. Taken any $x \in N$ means $\langle x, y \rangle = 0$. It will be shown that $T(x) \in N$. Note that the:
   $\langle T(x), y \rangle = \langle x, T^*(y) \rangle = \langle x, T(y) \rangle = \langle x, \theta \rangle = 0.$
   So, $T(x) \in N$. Thus, N invariant against T .
2. Because $\{x \in H : T(x) = \theta\}$ is the space the closed section H, then $N^{\perp} = \{x \in H : T(x) = \theta\}^{\perp\perp} = \{x \in H : T(x) = \theta\}.$
   Taken any $x \in N^{\perp}$ means $T(x) = \theta$. Note that the, $T(T(x)) = T(\theta) = \theta$. So, $T(x) \in N^{\perp}$. Thus, $N^{\perp}$ invariant against T.

According to 1 and 2, It can be concluded that N the reduction of T .

The following will be discussed the residual spectrum properties of linear operators on a Hilbert space is limited. It is understood that, in a given H Hilbert spaces and $T \in B(H)$, the residual spectrum dinotasikan $\sigma_r(T)$ is a compilation of all $\lambda \in C$ so $T_\lambda = T - \lambda I$ injective but $\overline{R(T_\lambda)} \neq H$ [5].

Theorem 12.Note H Hilbertspaces , $T \in B(H)$, and $\lambda \in C$. So $S_T(\lambda) = \varnothing$ and T_λ not surjective, maka $\lambda \in \sigma_r(T)$.

Proof: Based on the definition of, $S_T(\lambda)$ is the set of all spectral sequence - T_λ . Because $S_T(\lambda) = \varnothing$ then there are 2 possibilities i.e.:

1. $\|x_n\| = 1$ but $\lim_{n\to\infty} \|T(x_n) - \lambda(x_n)\| \neq 0$. Formed $\varepsilon = \inf_{x \in S_H} \|T(x) - \lambda x\|$ with $S_H = \{x \in H : \|x\| = 1\}$. Retrieved $\varepsilon > 0$. So,
   $\|T(x) - \lambda x\| \geq \varepsilon \|x\|, \forall x \in S_H$. It will be shown T_λ injective. Taken any $x_1, x_2 \in S_H$ with,
   $T_\lambda(x_1) = T_\lambda(x_2) \Leftrightarrow T_\lambda(x_1) - T_\lambda(x_1) = T_\lambda(x_1 - x_2) = \theta.$
   As a result, $\|T_\lambda(x_1 - x_2)\| = 0$. On the other hand,
   $\|T_\lambda(x_1 - x_2)\| \geq \varepsilon \|x_1 - x_2\|$
   $\Leftrightarrow 0 \geq \varepsilon \|x_1 - x_2\|$
   $\Leftrightarrow \|x_1 - x_2\| \leq 0.$
   Retrieved, $\|x_1 - x_2\| = 0$ .So,
   $x_1 - x_2 = \theta \Rightarrow x_1 = x_2$. Thus, T_λ injective.
2. $\|x_n\| \neq 1$ but $\lim_{n\to\infty} \|T(x_n) - \lambda(x_n)\| = 0$. It will be shown T_λ injective. Taken any $x_1, x_2 \in H$ with $\|x_1\|, \|x_2\| \neq 1$,
   $T_\lambda(x_1) = T_\lambda(x_2) \Leftrightarrow T_\lambda(x_1) - T_\lambda(x_1) = T_\lambda(x_1 - x_2) = \theta.$

As a result, $\|T_\lambda(x_1 - x_2)\| = 0$. So, must be $x_1 - x_2 = \theta$. So, $x_1 = x_2$. Thus, T_λ injective. Furthermore, it will be shown $R(T_\lambda)$ not dense in H. Suppose $R(T_\lambda)$ dense di Hmeans $\overline{R(T_\lambda)} = H$. T_λ notsurjective means there is $y_0 \in H$ so for each $x \in D(T_\lambda)$ apply $T_\lambda(x) \neq y_0$. Because $y_0 \in H$ and $\overline{R(T_\lambda)} = H$ so $y_0 \in \overline{R(T_\lambda)}$ means there is a sequence $(y_n) \subseteq R(T_\lambda)$ so $y_n \to y_0$. Note that for each $m, n \geq n_0$:

$\|x_n - x_m\| M \leq \|T_\lambda(x_n - x_m)\| = \|T_\lambda(x_n) - T_\lambda(x_m)\| = \|y_n - y_m\| \to 0.$

As a result, $\|x_n - x_m\| \to 0$. So, (x_n) is the Cauchy sequences in H. BecauseH then there are the complete $x \in H$ so $x_n \to x$. Because T_λ continuous and $x_n \to x$ so $T_\lambda(x_n) \to T_\lambda(x)$. With the uniqueness of the limit, retrieved $T_\lambda(x) = y_0$. This is a contradiction with the known $T_\lambda(x) \neq y_0$. The supposition is wrong, the truth is $R(T_\lambda)$ not dense in H. Thus, $\lambda \in \sigma_r(T)$.

After conveyed regarding the properties of the spectrum of a linear operator residual point and limited space, Hilbert would henceforth be discussed regarding the connecting point spectrum.

GivenH Hilbert spaces and $T \in B(H)$. Joint point spectrum, the notation $\sigma_{jp}(T)$, is the set of all $\lambda \in C$ so there is a $x \in H$ No scratch that meet $T(x) = \lambda x$ and $T^*(x) = \bar{\lambda} x$ [5].

Example 13. Given a Hilbert space H andN the closed section of the room H. Note

$N \neq \{\theta\}, N \neq H$, andP the projection operator N onH. For any $y \in N$, $y \neq \theta$ apply:

$P(y) = y = 1y$ and

$\langle P^*(y), y \rangle = \langle y, P(y) \rangle = \langle y, y \rangle$. So, $P^*(y) = 1y$. Thus, $1 \in \sigma_{jp}(P)$.

Theorem 14.NoteH Hilbert space and $T \in B(H)$. If for $0<p<\frac{1}{2}$, T p- hyponormal, so $\sigma_{jp}(T) = \sigma_p(T)$.

Proof: According To The Theorem 3.1.13, because $T \in B(H)$ then there is an operator of partial isometriUso $T = U|T|$, with $|T| = (T^*T)^{\frac{1}{2}}$.

1) Taken any $\lambda \in \sigma_{jp}(T)$ means there is a $x \in H$ No scratch that meet $T(x) = \lambda x$ and $T^*(x) = \overline{\lambda} x$. So, $\lambda \in \sigma_p(T)$.
2) Taken any $\lambda \in \sigma_p(T)$ means there is a $x \in H$ No scratch that meet $T(x) = \lambda x$. Enough shown $T^*(x) = \overline{\lambda} x$.

a. So $\lambda = 0$ then $T(x) = 0x = \theta$. Because $N(T) = N(T^*)$ then $T^*(x) = \theta$. So, $\lambda \in \sigma_{jp}(T)$.

b. So $\lambda \neq 0$ then

$$\tilde{T}(|T|^{\frac{1}{2}}(x)) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} |T|^{\frac{1}{2}}(x) = |T|^{\frac{1}{2}} U |T|(x) = |T|^{\frac{1}{2}} T(x) = |T|^{\frac{1}{2}} \lambda(x) = \lambda |T|^{\frac{1}{2}}(x)$$

and

$$\tilde{T}^*(|T|^{\frac{1}{2}}(x)) = |T|^{\frac{1}{2}} U^* |T|^{\frac{1}{2}} |T|^{\frac{1}{2}}(x) = |T|^{\frac{1}{2}} U^* |T|(x) = |T|^{\frac{1}{2}} |T| U^*(x)$$
$$= |T|^{\frac{1}{2}} T^*(x) = |T|^{\frac{1}{2}} \overline{\lambda}(x) = \overline{\lambda} |T|^{\frac{1}{2}}(x).$$

As a result, $U^*|T|(x) = \overline{\lambda}(x)$. So, retrieved $T^*(x) = |T| U^*(x) = U^*|T|(x) = \overline{\lambda}(x)$. Thus, $\lambda \in \sigma_{jp}(T)$. with Theorem 14, obtained the following result: Corollary 15[9]. NoteH Hilbert space andfor $0<p<\frac{1}{2}$. So T p- hiponormaland $\lambda \in \sigma_p(T)$, then $\overline{\lambda} \in \sigma_p(T^*)$.

Proof: Because $\lambda \in \sigma_p(T)$ then according to the theorem 14 $\lambda \in \sigma_{jp}(T)$ means there is a $x \in H$ No scratch that meet $T(x) = \lambda x$ and $T^*(x) = \overline{\lambda} x$. Because there is a $x \in H$ no scratch that meet $T^*(x) = \overline{\lambda} x$ then $\overline{\lambda} \in \sigma_p(T^*)$.

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