STUDY ON PLATE ELEMENT USING CLASSICAL THEORY AND FINITE ELEMENT METHOD

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Abstract- The fundamental obstacle in Structural Mechanics is the determination of deformational response of the structures subjected to different loading conditions. Various researchers have devised Classical theories for resolving the static and dynamic loading on structures. However, the Classical method of analyzing the structures become inconvenient under complicated boundary conditions. In order to overcome such complications the use of modern software, utilizing new methods like Finite Element Method (FEM), Matrix Method etc. have become quite frequent in the field of Structural Mechanics. This paper provides a comparative analysis of the results estimated by the Classical Plate Theory of Timoshenko and Krieger and those obtained by a Finite Element Method(FEM) based Software in case of a plate subjected to loading under different supporting conditions. This paper suggests the advantageousness and aptness of the FEM based software in estimating the response of plate element.

Keywords- Plate, Classical Theory of Plates, Finite Element Method, ANSYS, Plate with hole, Timoshenko and Krieger.

I. INTRODUCTION

Different structures are usually built up of beams, plates and shells-type structural elements.(Ref 1) Since these structural elements are important in so many of the modern structures, engineers and designers should have a good knowledge of their structural behaviour when these are subjected to static, dynamic and environmental loads.(Ref 2) In all these structural components, the basic problem is the determination of deformational response of the structures subjected to various loads.

The response of the structure subjected to various loads can be studied by the formulation of the Mathematical Models. Vast literature based on Classical Plate Theory exists for solving commonly used structural elements. [4], [5] But the classical theory is very cumbersome for the arbitrary shape and complex boundary conditions of structural elements such as plates and beams and cannot be formulated. Plates (Ref 3) with the holes at different locations is one of such cases and hence, classical solution of the problems is quite difficult and lengthy for them.

To overcome such complicated problems, some new techniques have evolved with the help of digital computers for the analysis of structures such as Finite Element Method (FEM), Matrix Method, etc. Now a days, a finite element method based software package like ANSYS 14.5 is being used to solve the problems of plates subjected to various loading.

It is not possible to obtain analytical solution for many engineering problems. An analytical solution is a mathematical expression that gives the value of the field variable at any location in the body. For the problems involving complex shapes, material properties and complicated boundary conditions, it is difficult to and in many cases intractable to obtain analytical solutions that satisfies the governing differential equations. Hence, for most of the practical problems, engineers resort to numerical methods that provide approximate but acceptable solutions. Finite Element Method (FEM) is one of them. Work of Turner, Clough, Martin and Topp (1956) on stiffness and deflection analysis of complex structures, led to the discovery of Finite Element Method.(Ref 6) Clough (1960) in his subsequent work on the finite element plane stress analysis gave the physical interpretation to the method. This paper gives us insight into the solution of a plate with and without hole and compares the result obtained by the numerical method software namely ANSYS 14.5 with that obtained through Classical plate theory given by S. Timoshenko and S.W. Krieger.(Ref 7)

II. METHODOLOGY

2.1. Analytical Expressions

The analytical expressions for the boundary conditions are formulated in the following discussion.

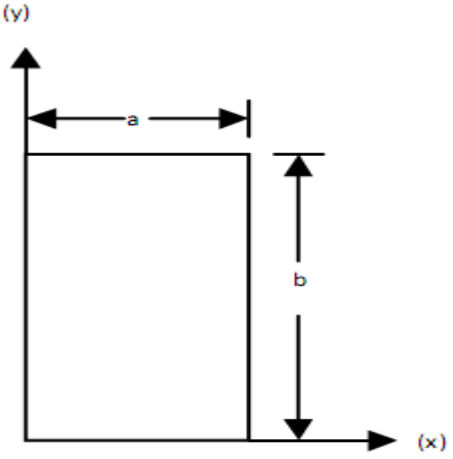


Fig 1: A Plane Plate

2.1.1. Simply Supported Edge

If the edge y=0 of the plate is simply supported, the deflection w along this edge must be zero. At the

same time this edge can rotate freely with respect to the x-axis; that is, the analytical formulation of the physical boundary condition is (Ref 5) (Ref 9)

$(w)_{y=0}=0$ and $\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0}=0$

Similarly, on the edge x=0

$(w)_{x=0}=0$ and $\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0}=0$

2.1.2. Clamped or Built-in Edges

If an edge of a plate is clamped, the deflection w along this edge must be zero. At the same time this edge cannot rotate about the support.

If the edge y=0 of the plate is clamped then,

$(w)_{y=0}=0$ and $\left(\frac{\partial w}{\partial y}\right)_{y=0}=0$

Similarly, on the edge x=0

$(w)_{x=0}=0$ and $\left(\frac{\partial w}{\partial x}\right)_{x=0}=0$

2.2. Linear Theory of Plates

From above general theory, for small deflection problem i.e., for linear case in which the deflection is small in comparison with the plate thickness, $\varphi = 0$

$$D\left[\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right]w=q$$

2.2.1. Simply supported and uniformly loaded rectangular plate (Ref 7)

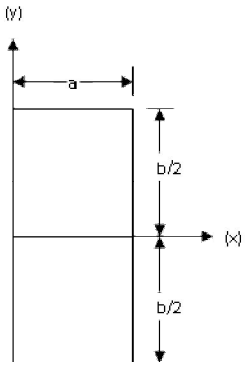


Fig 2: Simply supported and uniformly loaded rectangular plate

$$w=\sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a}$$ (Ref 10)

Where Y_m is a function of 'y'only.

At x = 0, and x = a

w =0 and $\left(\frac{\partial^2 w}{\partial x^2}\right)=0$

Y_m should be in such a form as to satisfy the boundary conditions on the sides y = ± b/2 and also the equation of the deflection surface

$$\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]=\frac{q}{D} \qquad \text{(a)}$$

The deflection surface is then represented by

$$w=\frac{q}{24D}\left(x^4-2ax^3+a^3x\right)+\frac{qa^4}{D}\sum_{m=1}^{\infty}\left(A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}$$

which satisfies eq. (a) and also boundary conditions at x =0 & x=a.

The maximum deflection at the Centre of the plate (x = a/2, y = 0)

$$w_{\max}=\frac{5}{384}\frac{qa^4}{D}-\frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5....}^{\infty}\frac{(-1)^{(m-1)/2}}{m^5}\frac{\alpha_m\tanh\alpha_m+2}{2\cosh\alpha_m}$$

$$w_{\max}=\alpha\frac{qa^4}{D}$$

Along the x-axis the expression for the bending moments

$$(M_x)_{y=0}=\frac{qx(a-x)}{2}-qa^2\pi^2\sum_{m=1,3,5.....}^{\infty}m^2\left[2\nu B_m-(1-\nu)A_m\right]\sin\frac{m\pi x}{a}$$

$$(M_y)_{y=0}=\nu\frac{qx(a-x)}{2}-qa^2\pi^2\sum_{m=1,3,5.....}^{\infty}m^2\left[2B_m+(1-\nu)A_m\right]\sin\frac{m\pi x}{a}$$

Maximum values of these moments at the centre of the plat (x = a/2, y = 0)

$(M_x)_{\max}=\beta qa^2 \quad (M_y)_{\max}=\beta_1 qa^2$

Numerical values of α, β, β_1 for different aspect ratios are given in Ref 7

2.2.2 Rectangular plate with two opposite edges Simply supported and the other two edges Clamped (Ref 7)

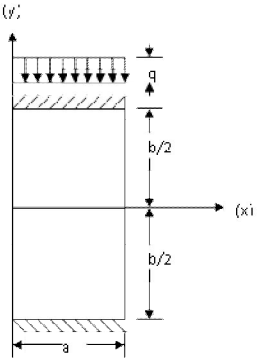


Fig 3: Rectangular Plates with two opposite edges Simply Supported and the other two edges Clamped

Assume that the edges x=0 and x=a of the rectangular plate are simply supported and other two edges are clamped. (Ref 11)

The deflection at the centre of the plate can be represented by the formula

$$w=\alpha\frac{qa^4}{D}$$

The values of moments at the centre of the plate can be calculated as:

$$M_x=\frac{2qa^2}{\pi^3}\sum_{m=1,3,5.....}^{\infty}\frac{(-1)^{(m-1)/2}}{m^3\cosh\alpha_m}\frac{\alpha_m-\tanh\alpha_m(1+\alpha_m\tanh\alpha_m)}{\alpha_m-\tanh\alpha_m(\alpha_m\tanh\alpha_m-1)}\left[2\nu+(1-\nu)\alpha_m\tanh\alpha_m\right]$$

$$M_y=\frac{2qa^2}{\pi^3}\sum_{m=1,3,5.....}^{\infty}\frac{(-1)^{(m-1)/2}}{m^3\cosh\alpha_m}\frac{\alpha_m-\tanh\alpha_m(1+\alpha_m\tanh\alpha_m)}{\alpha_m-\tanh\alpha_m(\alpha_m\tanh\alpha_m-1)}\left[2-(1-\nu)\alpha_m\tanh\alpha_m\right]$$

The final values of the moments at the centre of the plate can be represented by

$M_x=\beta_1 qa^2 \quad M_y=\beta_2 qa^2$

Numerical values of α, β_1 β_2 for different aspect ratios are given in (Ref 7)

2.3. Finite Element Formulation: Derivation of the Element Stiffness Matrix

The element stiffness matrix is derived here by applying the Principle of Minimum Potential Energy. The potential energy of the plate element is given by (Ref 12)

$$\phi = \frac{1}{2}\int_0^a\int_0^b\left(-M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)dxdy - \int_0^a\int_0^b\{f\}^T q dxdy$$

- (a)

Where q is any discrete loading inside the element. Based on the notation used for the rectangular plate bending element, eqn. (a) can be written as

$$\phi = \frac{1}{2}\int_0^a\int_0^b\{\varepsilon\}^T\{\sigma\}dxdy - \int_0^a\int_0^b w^T q dxdy \quad \text{(b)}$$

$$\{f\} = w = [N]\{X\}_e \quad \text{(c)}$$

$$\{\varepsilon\} = [B]\{X\}_e \quad \text{(d)}$$

$$\sigma = [D][B]\{X\}_e \quad \text{(e)}$$

Combining equations (b), (c), (d) and (e), yields

$$\phi = \frac{1}{2}\int_0^a\int_0^b\{X\}_e^T[B]^T[D][B]\{X\}_e dxdy - \int_0^a\int_0^b\{X\}_e^T[N]^T q dxdy \quad \text{(f)}$$

The principle of minimum potential energy requires

$$\left\{\frac{\partial\phi}{\partial\{X\}_e}\right\} = \{0\}$$

Therefore, performing the partial differentiation of equation (f), yields

$$\left\{\frac{\partial\phi}{\partial\{X\}_e}\right\} = \int_0^a\int_0^b[B]^T[D][B]dxdy\{X\}_e - \int_0^a\int_0^b[N]^T q dxdy = \{0\}$$

Or

$$[K]_e\{X\}_e = \{P\}_e$$

Where

$$[K]_e = \int_0^a\int_0^b[B]^T[D][B]dxdy$$

$$\{P\}_e = \int_0^a\int_0^b[N]^T q dxdy$$

III. RESULTS

In the cases of Simply Supported rectangular plate, and rectangular plate with two opposite edges clamped and other two Simply Supported, the responses corresponding to the load application have been tabulated. Moreover, the responses of rectangular plate with circular hole obtained through Finite Element based Software, i.e. ANSYS 14.5 has been tabulated.

Following are the common properties for all the cases of plates:

Length in x- direction= a

Length in y-direction= b

Poisson's ratio, υ=0.3

Uniformly distributed load intensity=5.88×10^{-3} N/mm^2

For Plain Plate

Plate thickness, h=120 mm

Young's modulus of elasticity, E=2×10^4 N/mm^2

3.1. Rectangular plate with all edges Simply Supported

S. No.	Aspect ratio	Analytical solution (Timoshenko)			Finite element solution		
	a/b	W_{max} (mm)	Mx_{max} (N-mm)	My_{max} (N-mm)	W_{max} (mm)	Mx_{max} (N-mm)	My_{max} (N-mm)
1	1.0	4.71E-4	70.41	70.41	4.68E-4	71.15	71.15
2	0.833	3.16E-4	64.01	51.14	3.14E-4	64.53	51.78
3	0.667	1.77E-4	53.05	32.54	1.76E-4	53.33	33.06
4	0.5	7.35E-5	37.37	17.05	7.32E-5	37.49	17.43
5	0.333	1.75E-5	19.42	6.63	1.75E-5	19.41	7.55

Table 1: Response recorded for Rectangular plate with all edges simply Supported

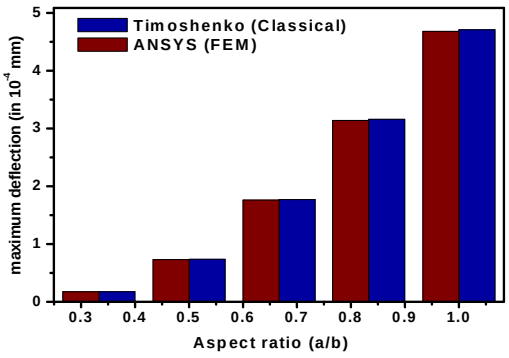


Fig 4(a): Comparison of maximum deflection between Classical method and FEM

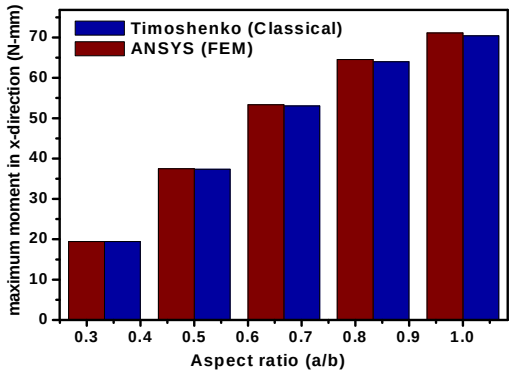


Fig 4(b): Comparison of maximum moment in x-direction between Classical method and FEM

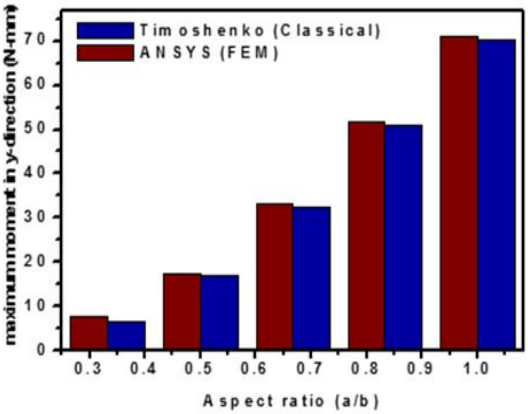


Fig 4(c): Comparison of maximum moment in y-direction between Classical method and FEM

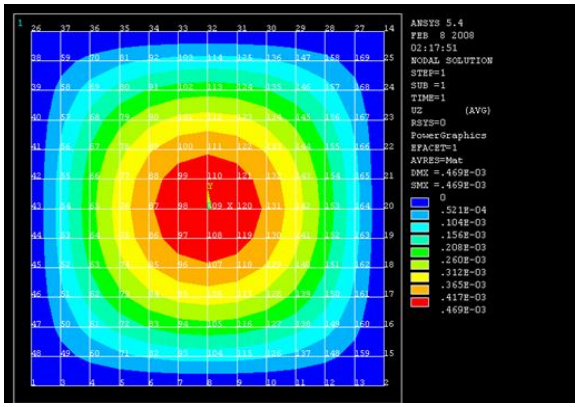


Fig 5(a): Deflection in Z-direction for square plate

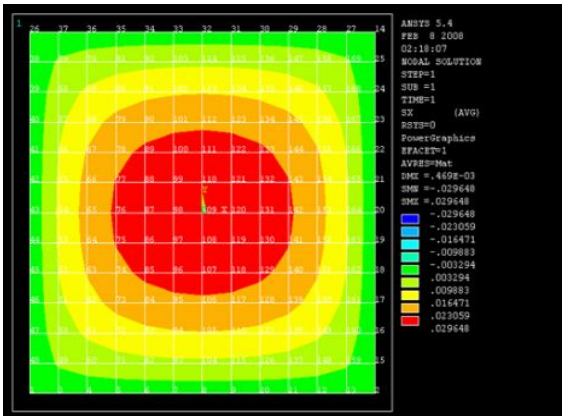


Fig 5(b): Stress in x-direction for square plate

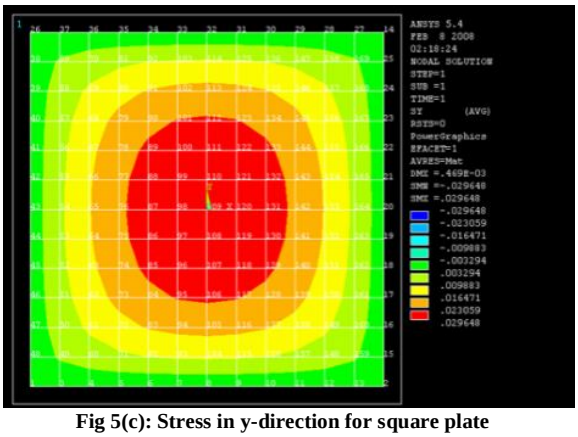


Fig 5(c): Stress in y-direction for square plate

3.2. Rectangular plate with two opposite edges clamped and other two Simply Supported

S. No.	Aspect ratio	Analytical solution(Timoshenko) (at centre)			Finite element solution (at centre)		
	a/b	Wmax (mm)	Mx (N-mm)	My (N-mm)	Wmax (mm)	Mx (N-mm)	My (N-mm)
1	1.0	2.23 E-4	35.87	48.80	2.23E-4	36.81	49.85
2	1.2	2.59 E-4	31.61	55.13	2.60E-4	32.64	56.00
3	1.4	2.79 E-4	28.22	58.65	2.82E-4	28.68	59.52
4	1.5	2.87 E-4	26.31	59.68	2.89E-4	26.96	60.61
5	2.0	3.02 E-4	20.87	61.74	3.07E-4	21.28	62.60

Table 2: Response recorded for rectangular plate with two opposite edges clamped and other two simply supported

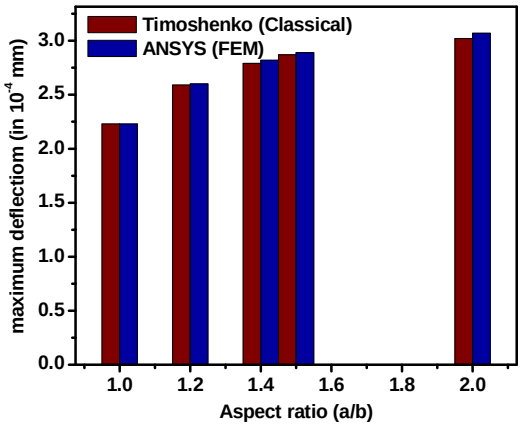


Fig 6(a):Comparison of maximum deflection between Classical method and FEM

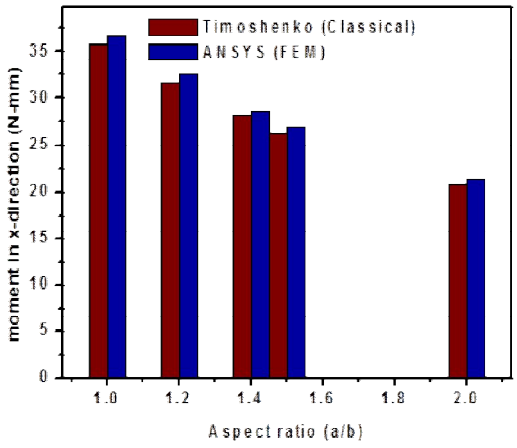


Fig 6(b):Comparison of moment in x-direction at centre between Classical method and FEM

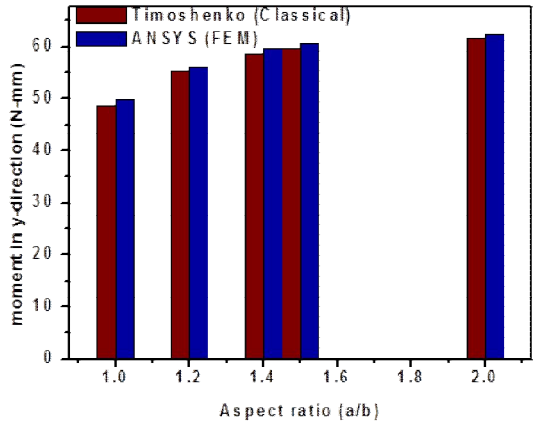


Fig 6(c):Comparison of moment in y-direction at centre between Classical method and FEM

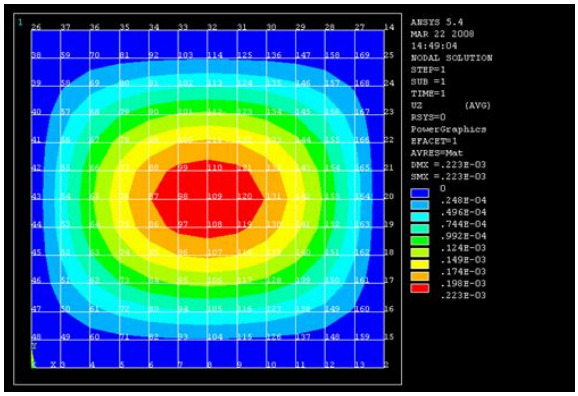


Fig 7(a):Deflection in Z-direction for square plate

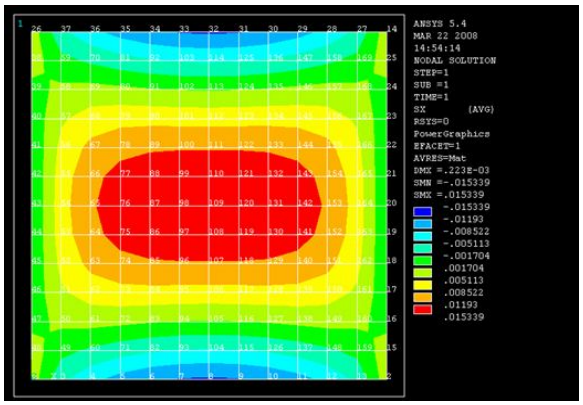


Fig 7(b):Stress in x-direction for square plate

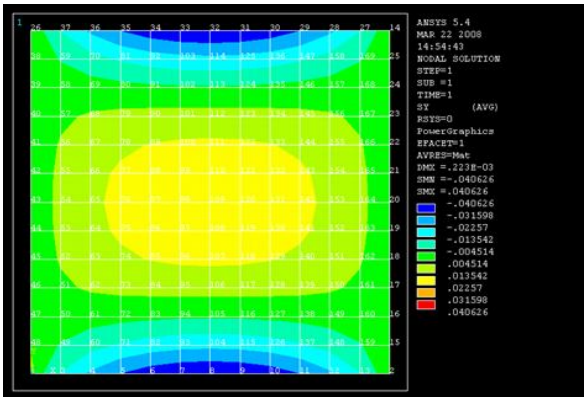


Fig 7(c) :Stress in y-direction for square plate

3.1. Rectangular plate with two opposite edges free and other two simply supported and a circular hole at centre

S. No.	Aspect ratio	Finite element solution		
	a/b	W_{max} (mm)	Mx_{max} (N-mm)	My_{max} (N-mm)
1	1.0	1.95E-3	53.06	307.15
2	1.5	1.86E-3	69.40	300.58
3	2.0	1.85E-3	77.78	297.72
4	2.5	1.85E-3	85.12	300.65
5	3.0	1.84E-3	86.96	302.95

Table 3: Response recorded for rectangular plate withtwo opposite edges free and other two simply supported and a circular hole at centre

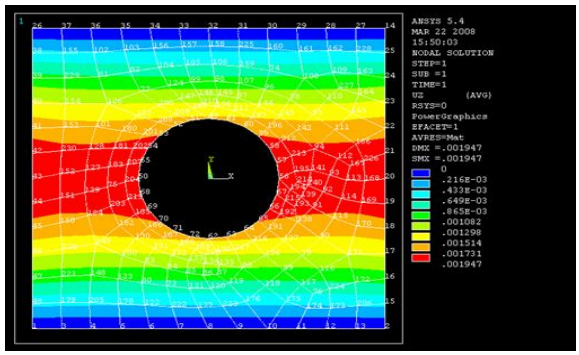


Fig 8(a): Deflection in Z-direction for square plate

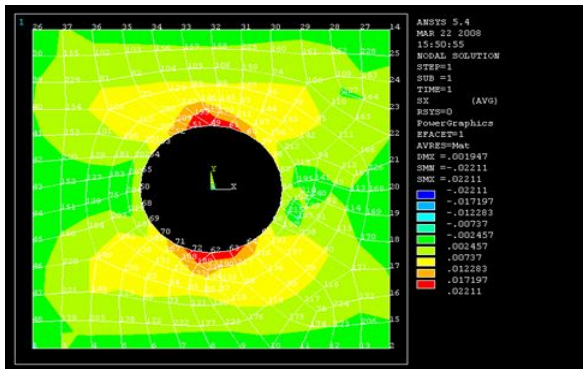


Fig 8(b): Stress in x-direction for square plate

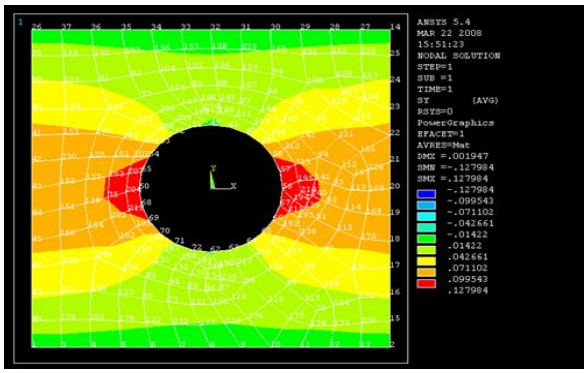


Fig 8(c): Stress in y-direction for square plate

DISCUSSIONS

In the case of simply supported rectangular plate and rectangular plate with two opposite edges clamped and other two simply supported, maximum deflection (W_{max}), maximum moment in X-direction (Mx_{max}) and maximum moment in Y-direction (My_{max}) was computed by using Classical Theory of Plates by Timoshenko and Krieger (Ref 5) and then by using FEM-based computing software, namely ANSYS 14.5 corresponding to various aspect ratios.

The comparative analysis of the results obtained by Conventional Theory(Classical Method) and FEM-based theory leads to the development that, for different values of aspect ratio, the values of the maximum deflection (W_{max}), the maximum moment in x-direction(Mx_{max}) and the maximum moment in y-direction(My_{max}) were almost same with infinitesimally variation of 0.63% in the value if ever it was existing, in the case of simply supported rectangular plate (Fig 4(a), 4(b) and 4(c)). Further, the case of rectangular plate with two opposite edges clamped and other two simply supported just complemented the findings with maximum variation of 1.65% (Fig 6(a), 6(b) and 6(c)).

The resultant deflection , stresses in X-direction and stresses in Z-direction were displayed comprehensively in the case of two plates with aforesaid support condition with the help of FEM based software, i.e. ,ANSYS 14.5 (Fig 5(a), 5(b) and 5(c)),(Fig 7(a), 7(b) and 7(c)).

Moreover, the FEM based Software was efficient and effective in estimation of Maximum Deflection, Stresses in x-direction and stress in y-direction in case of a Rectangular plate with two opposite edges free and other two simply supported and a circular hole at centre. (Fig 8(a),8(b) and 8(c))The solutions could not have been easily arrived at by adopting Classical Theory due to complexity of the structure.

CONCLUSIONS

It was quite evident from this research that the responses yielded by the plates under different supporting conditions computed by FEM based Software, namely ANSYS 14.5 are in complete agreement with the Classical Theory of Plates given by Timoshenko and Krieger.

In addition to that, Finite Element Method is advantageous over Classical Plate Theory as it could be applied even to the Complex structures with unrestricted boundary conditions as was noticeable in the case of rectangular plate with central circular hole. So, there is neither any geometric restriction nor any loading constraint on the application of FEM-based software.

A critical look at the existing literature thus emphasizes the need and scope for research in understanding the linear and nonlinear behavior of plates subjected to static and dynamic loading conditions.

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