TWO-DIMENSIONAL SLOW STAGNATION FLOW WITH SHIFTED CENTERLINE TOWARD A SLIT

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Abstract- Two-dimensional slow stagnation flow having a shifted centerline toward a slit is investigated on the basis of the Stokes approximation. The flow field is expressed in terms of two complex analytic functions. The functions which make the flow satisfy the boundary conditions are analytically determined from the solutions of a pair of Riemann-Hilbert problems. The streamline patterns including the local behavior near the sharp edges and the stress distributions on the plates are calculated. The effects of the distance of centerline shift on the characteristics of flow including the shear stress on the plate are also discussed.

Index Terms- Centerline Shift, Complex Velocity, Stagnation Flow, Stokes Approximation.

I. INTRODUCTION

On any solid body placed in a moving fluid, there exists a stagnation point on the front side of it. The stagnation point flow toward a plane wall has attracted much attention in association with the flow approaching a blunt body or a finite flat plate [1]. The two-dimensional stagnation point flow, usually called the Hiemenz flow [2], is one of the rare exact solutions of Navier-Stokes equations and regarded as an important flow since many flows near a stagnation point of a solid body can be approximated by it. White [3] gives an extensive discussion on the features of stagnation point flow in association with boundary layer flows. There have been several studies on the modified version of the Hiemenz flow. The stagnation flow around a vertical plate attached on a plane wall was considered numerically by Suh and Liu [4] for the Reynolds numbers up to 2800 [4], and analytically by Jeong and Ko [5] based on the Stokes approximation. Jeong [6] further developed the latter study such that the vertical plate was allowed to detach from the plane wall. Ko and Jeong [7] studied the two-dimensional slow stagnation flow towards a plate with a slit based on the Stokes approximation.

There have been several studies on the two-dimensional Stokes flows around a slit. A flow due to a Stokeslet near a slit was investigated by Ko and Kim [8] as a first approximation of the viscous flow due to the translation of a cylindrical body with small diameter. In a study on the flow past a thin screen, Hasimoto [9] also investigated the flow through a slit which is generated by a pressure difference by finding a solution of the associated potential flow. As a kind of the generalized version of this flow, Jeong and Kim [10] studied the flow in a converging nozzle with an arbitrary converging angle. In this study, two-dimensional slow stagnation flow having a shifted centerline toward a slit is investigated. This flow can be realized in the vicinity of a crack of finite plate in a fluid in motion. Because it is uncommon that the dividing stream line exactly attaches to the crack, it can be regarded as a generalization of the flow of [7] in that the centerline of the approaching flow and that of the slit are allowed to be some distance apart. As the inertia effect is small near a plate with a crack, the Stokes approximation is assumed to be valid in analyzing the flow. Then the flow can be described by two complex analytic functions which are analytically determined from the solution of a pair of Riemann-Hilbert problems resulting from the boundary conditions. Using the closed form solution, the streamline patterns including the local behavior near the sharp edges, and the pressure and the shear stress distributions on the plates are calculated. The effects of the distance of centerline shift on the flow are also discussed.

II. FORMULATION AND ANALYSIS

As shown in Fig. 1, two semi-infinite horizontal plates form a slit having a dimensionless length of 2 and the origin of the two-dimensional Cartesian coordinates (x, y) coincides with the center of the slit.

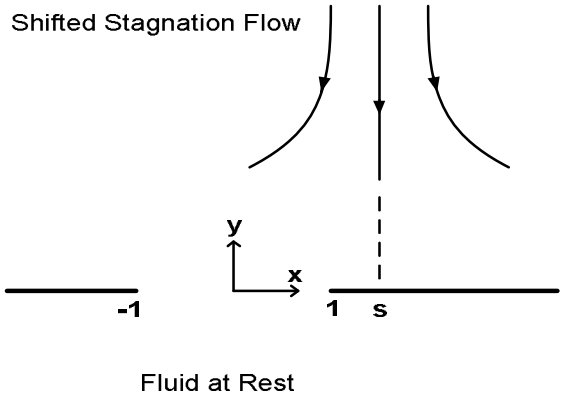


Fig. 1. Schematic illustration of the flow problem.

A viscous flow due to a slow stagnation flow from infinity above the plates with a strength of K approaching the slit is investigated. The centerline of the stagnation flow is shifted from that of the slit (y axis) to the right by a distance of s and the fluids in the lower half-plane far from the slit are at rest.

The governing equations for the viscous flow under the Stokes approximation are expressed as

$$\mu\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}\right)=\frac{\partial p}{\partial x}, \quad (1)$$

$$\mu\left(\frac{\partial^2 v}{\partial^2 x}+\frac{\partial^2 v}{\partial^2 y}\right)=\frac{\partial p}{\partial y}, \quad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad (3)$$

Here μ is the viscosity of the fluid, and u, v, and p denote the x and y components of velocity, and the pressure of the fluid, respectively.

As is already known, using imaginary unit $i=\sqrt{-1}$ and introducing complex coordinate and complex velocity as

$$z=x+iy, \quad (4)$$

$$W=u-iv, \quad (5)$$

the solution to the system of (1)-(3) can be represented using two complex analytic functions as follows [11]:

$$W=F(z)+(\bar{z}-z)\frac{dG}{dz}-\overline{G}(\bar{z}), \quad (6)$$

$$p-i\mu\Omega=4\mu\frac{dG}{dz}. \quad (7)$$

Here overbar denotes the complex conjugate and Ω is the vorticity of the fluid. The complex functions $F(z)$ and $G(z)$ must be analytic in the whole z-plane except the plates. Since the flow region is symmetric with respect to x axis, the conjugate function $\overline{G}(z)=\overline{G(\bar{z})}$ is also analytic in the whole flow region owing to the Schwarz reflection principle [12].

The boundary conditions for the velocity are no-slip condition at the plates (solid wall) and the conditions at the up- and down- infinities. These are as follows:

$$W=0 \ \ at \ \ y=0, \ |x|\geq 1, \quad (8)$$

$$W\rightarrow\begin{cases} Ky(2x-2s+iy) & as \ \ y\rightarrow\infty \\ 0 & as \ \ y\rightarrow-\infty \end{cases}. \quad (9)$$

In order to find the analytic functions F and G satisfying (8) and (9) via (6), the limiting behaviors of these functions on the upper and lower sides of the plates are to be examined. If we denote the limiting value of F or G on the upper/lower side by the subscript +/- and (6) is substituted, (8) can be rewritten for the limiting values as

$$[F]_+-[\overline{G}]_-=0, \quad (10)$$

$$[F]_--[\overline{G}]_+=0. \quad (11)$$

Subtraction and addition of (10) and (11) give the following relations between the boundary values of two new functions.

$$[F+\overline{G}]_+-[F+\overline{G}]_-=0, \quad (12)$$

$$[F-\overline{G}]_++[F-\overline{G}]_-=0. \quad (13)$$

Since the sum and the difference of analytic functions are also analytic, (12) and (13) are the Riemann-Hilbert problems for the analytic functions $F(z)+\overline{G}(z)$ and $F(z)-\overline{G}(z)$, respectively. Considering the condition (9), $F+\overline{G}$ and $F-\overline{G}$ should not grow faster than quadratic polynomial of z at infinity. An additional condition imposed on $F-\overline{G}$ is that the velocity should be finite at the sharp edges $z=\pm1$. This condition requires that the solution with the lowest singularity at the sharp edges should be selected among all possible solutions. Following the standard solution procedure for the Riemann-Hilbert problem, a set of general solutions of (12) and (13) compatible with these conditions are expressed as follows [12]:

$$F(z)+\overline{G}(z)=A_0+A_1z+A_2z^2, \quad (14)$$

$$F(z)-\overline{G}(z)=\sqrt{z^2-1}\left(B_0+B_1z\right). \quad (15)$$

Here the unknown complex constants A_0, A_1, A_2, B_0, and B_1 are determined such that the condition (9) is satisfied. The double valued function $\sqrt{z^2-1}$ can be made single valued by restricting the arguments as

$$-\pi<\arg(z+1)<\pi, \ \ 0<\arg(z-1)<2\pi. \quad (16)$$

This choice is identical to taking two flat plates as the branch cuts of $\sqrt{z^2-1}$, which is reasonable in a physical point of view. When the arguments are determined in this way, it is to be noted that the conjugate function of $\sqrt{z^2-1}$ is equal to $-\sqrt{z^2-1}$. Solving (14) and (15) and utilizing this relation, we can calculate W in a form including the unknown constants. Then the constants which make the condition (9) is satisfied are easily calculated. The final solutions are expressed as follows:

$$F(z)=-\frac{iK}{8}(z-2s)\left(z+\sqrt{z^2-1}\right), \quad (17)$$

$$G(z)=\frac{iK}{8}(z-2s)\left(z+\sqrt{z^2-1}\right). \quad (18)$$

As we have determined two analytic functions describing the flow field, we can calculate the distributions of velocity, pressure, and vorticity, and other physical quantities of the flow from (6) and the like.

RESULTS AND DISCUSSIONS

Two-dimension slow stagnation flow with shifted centerline toward a slit can be described in a closed form by using $F(z)$ and $G(z)$. The streamline patterns of the two-dimensional flows are easily described in terms of the stream function which automatically satisfies the continuity equation (3). The general form of the conventional stream function □□compatible with W in (6) is written as follows [11]:

$$\Psi = \mathrm{Im}\left[\int F(z) + G(z)dz + (\bar{z} - z)G(z)\right], \qquad (19)$$

where "Im" means the "imaginary part." Substituting (17) and (18) into (3) and (19), we obtain the following expressions for the complex velocity and the stream function:

$$W = (K/4)\left\{y(z + \sqrt{z^2 - 1})(z - 2s + \sqrt{z^2 - 1})/\sqrt{z^2 - 1} + \mathrm{Im}\left[(z - 2s)(z + \sqrt{z^2 - 1})\right]\right\}, \qquad (20)$$

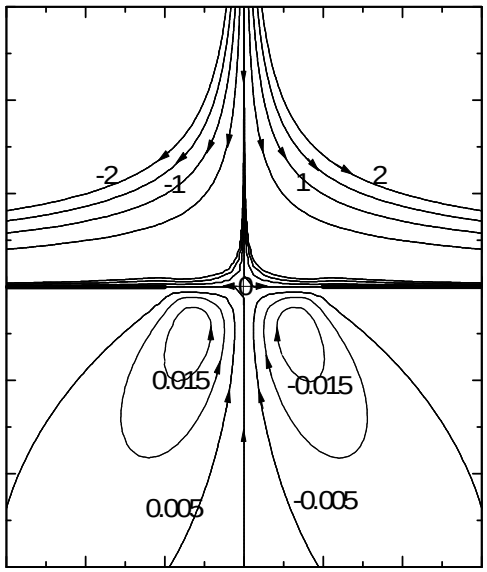
$$\Psi = (Ky/4)\mathrm{Im}\left[(z - 2s)(z + \sqrt{z^2 - 1})\right]. \qquad (21)$$

It can be easily known that □□□= 0 on the horizontal line $y = 0$ (x- axis) for any values of K and s. This implies that the volume flowrate through the slit is zero irrespective of the centerline shift.

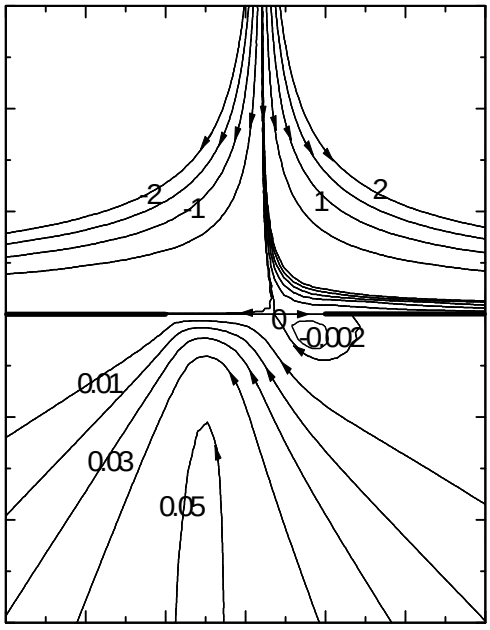
The streamline patterns obtained by numerically calculating □□ are shown in Figs. 2 (a)-(d) for the representative values of s in a domain of [-3, 3] x [-3, 3]. The numbers denote the values of the stream function which are calculated by taking $K = 1$. Because the flow region is symmetric with respect to the y axis and the system of governing equations of the flow, (1)-(3), is linear, only the cases for the nonnegative values of s are considered. The flow patterns for the negative values of s can be obtained by considering that of the positive s of the same absolute value and the symmetry. For example, the streamlines for $s = -0.2$ can be constructed from the mirror images of Fig. 2 (b), by a mirror placed at the line $x = 0$. Since the velocity scales are far different in the upper and lower half planes, the incremental values of □□ between adjacent streamlines are taken to be different in each region.

The flow pattern for the case without centerline shift is as shown in Fig. 2 (a), which is in agreement with the result of [7]. The saddle stagnation point exists at the center of the slit, that is, at the origin (0, 0). The stagnation streamlines consist a vertical line (y axis) and a horizontal line (x axis). A symmetric pair of viscous eddies appear in the lower half plane and extends to the lower infinity and the whole flow field shows a symmetry with respect to the y axis.

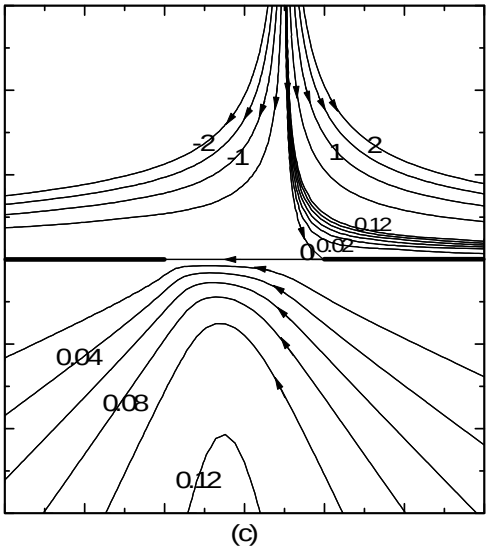
The symmetry of flow no longer exists in the other cases for nonzero values of s which are shown in Fig. 2 (b)-(d). In the first nonzero case of $s = 0.2$, the saddle stagnation point moves to the point (0.4, 0). One of the stagnation streamline remains a horizontal line, but the other attaches to the lower side of the right plate and encloses the eddy. There exist only one eddy of a small size. The flow in the lower half plane outside the eddy is similar to a flow which is swept by a simple shear flow in the upper half plane.



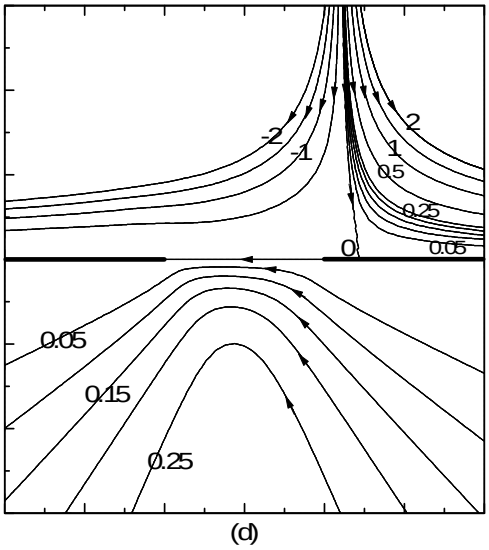
(a)



(b)



(c)



(d)

Fig. 2. Streamlines for (a) s = 0, (b) s = 0.2, (c) s = 0.5, and (d) s = 1.2. The numbers denote the values of the stream function which are calculated by taking K = 1.

The flow shown in Fig. 2 (c) is a flow which plays a dividing ridge of the two flow groups represented by 2 (b) and 2 (d) respectively. In this flow the eddy of 2 (b) shrinks to a point and the saddle stagnation points on the slit and at the lower side of the right plate merge to one point (1, 0) at the edge the right plate. The flow pattern in the lower region is very similar in shape to that of 2 (b), although the former is closer to a symmetric flow than the latter.

The last case shown in Fig. 2 (d) is a representative case of the flows with s larger than 1. The flows in the upper and lower half planes show more disconnected behavior. The stagnation streamline exists only in the upper region and attaches to a point in the right side of the edge. The flow in the lower region is nearly symmetric with respect to a certain vertical line.

In view of the streamline patterns in Fig. 2, the locations of the stagnation points and the separation (or attachment) of the flow from (or to) the plate more or less characterize the flow. First it is investigated whether the flow separates at the sharp edges of the plate. As explained, from symmetry, it is sufficient to analyze the edge of the right plate as long as s is allowed to be negative. Let us introduce a local polar coordinate (R, □) defined by

$$z = 1 + Re^{i\phi},\ (R \to 0,\ 0 < \phi < 2\pi)\ . \qquad (22)$$

After a lengthy calculation, the asymptotic behavior of the stream function near z = 1 is written as follows.

$$\psi \approx \left(K/\sqrt{2}\right)(1-2s)R^{3/2}\cos(\phi/2)\sin^2(\phi/2)\ . \qquad (23)$$

This form is in agreement with the expression about the general behavior of the Stokes flow near a sharp edge by Michael and O'Neill [13]. From (23) we can know that the flow separates from the sharp edge (z = 1) horizontally, that is, with a separation angle of □□□□□. The behavior near the left sharp edge z = -1 is obtained from (23) by replacing s with –s.

Finding the eddy centers like those in Fig. 2 (a), (b) is not straightforward. The coordinate transform method of [7] is very useful in finding zeros of W. Using this method and after a tedious calculation, the location of the eddy centers which exist for -0.5 < s < 0.5 can be expressed as follows:

$$z_{eddy} = \begin{cases} \dfrac{-2+2s}{3} - \sqrt{\dfrac{2(1+2s)^3}{9(1-4s)}}\,i & (-0.5 \le s \le 0) \\ \dfrac{2+2s}{3} - \sqrt{\dfrac{2(1-2s)^3}{9(1+4s)}}\,i & (0 \le s \le 0.5) \end{cases} \qquad (24)$$

When the flow is symmetric with respect to the y axis, with s = 0, a pair of viscous eddies exists having centers in the lower half plane on (-0.667, -0.471) and (0.667, -0.471), respectively (See Fig. 2 (a)). For nonzero values of s, there exists only one eddy in the third or fourth quadrant. As the absolute value of s increases up to 0.5, the eddy center approaches (-1, 0) or (1, 0) corresponding to the limiting cases of s = -0.5 or 0.5.

From (7) and (18) the pressure and the vorticity distributions are expressed as follows.

$$p - i\mu\Omega = \frac{i\mu K}{2}\left(z + \sqrt{z^2-1}\right)\left(1 + \frac{z-2s}{\sqrt{z^2-1}}\right). \qquad (25)$$

In order to locate the stagnation points on the plates or on the slit, it is necessary to calculate the velocity and the shear stress distributions for y = 0. Calculating W of (20) for y = 0, it follows that

$$W = \begin{cases} (K/4)(x-2s)\sqrt{1-x^2} & (-1 < x < 1) \\ 0 & (|x| > 1) \end{cases}. \qquad (26)$$

From (26), the saddle stagnation on the slit is easily known to locate at $x = 2s$. In addition it can be concluded that □□□= -□□ for $y = 0$ since $v = 0$ there. Therefore the final forms of the pressure and the shear stress distribution on the right plate are

$$p(x, \pm 0) = 0, \qquad (27)$$

$$\tau(x, \pm 0) = \frac{\mu K}{2}\left(x \pm \sqrt{x^2-1}\right)\left(1 \pm \frac{x-2s}{\sqrt{x^2-1}}\right). \qquad (28)$$

The stress distributions on the left plate can be easily obtained by using the symmetry of the flow. It is interesting that the pressure is constant (with 0) on both sides of the plate for any values of s. This behavior was found for the special case of $s = 0$ in [7]. The shear stress distributions on the right plate are calculated numerically making dimensionless by dividing by □K. They are shown for the upper side in Fig. 3 and for the lower side in Fig. 4, respectively.

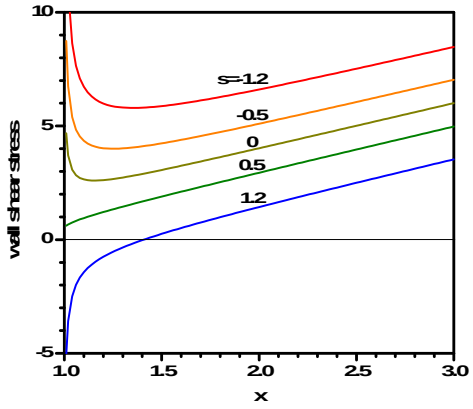


Fig. 3. The variation of the shear stress on the upper side of the right plate for some representative values of *s*. It is made dimensionless by dividing by □*K*.

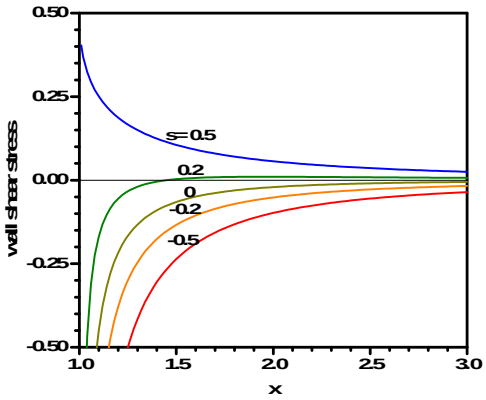


Fig. 4. The variation of the shear stress on the lower side of the right plate for some representative values of *s*. It is made dimensionless by dividing by □*K*.

The shear stress on the upper side of the right plate is larger for smaller s, and shows a decreasing-increasing behavior for $s < 0.5$ and simply increasing behavior for $s > 0.5$ with respect to x. Except for the case of $s = 0.5$, its magnitude grows to infinity as the sharp edge $x = 1$ is approached. If $s > 0.5$, there exists a point at which the shear stress is zero (See the graph in Fig. 3 for $s = 1.2$). This point from which the flow separates can be determined by examining (27) and is given by $x = s + 0.25/s$.

The shear stress on the lower side of the right plate shows a monotonically increasing dependence on s. Its dependence on x is also monotonic and shows an unbounded behavior at $x = 1$ unless $s = 0.5$. There exists a point at which the shear stress changes signs if $0 < s < 0.5$. For a sample case of $s = 0.2$, see Fig. 4. This is a flow separation point and its location is given by $x = s + 0.25/s$.

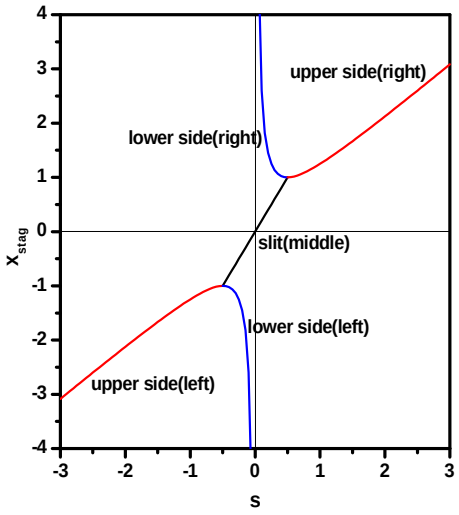


Fig. 5. The variation of the *x* component of the stagnation points with respect to the centerline shift *s*.

Looking into the flow patterns shown in Fig. 2, it can be found that there are two kinds of stagnation points: one is a saddle stagnation completely in the fluid and the other is a stagnation point attached to a solid wall. The former is formed on the slit when the centerline shift of the approaching stagnation flow is small, or more specifically for $-0.5 < s < 0.5$. One of the dividing stagnation streamlines is a horizontal line $y = 0$, and the other extends to the infinity or attaches to the lower side of the plate depending on the sign of s. For the values of s less than -0.5 or larger than 0.5, saddle stagnation point no longer exists. Only one stagnation point is formed on the upper side of the left plate ($s < 0$) or the right plate ($s > 0$). The streamline patterns are topologically same as those of the stagnation flow toward a plate without a slit.

Fig. 5 illustrates the features of the flow field using the dependence of the location of the stagnation points on

the centerline shift. All stagnation points locate on a horizontal line, $y = 0$. The x component of the stagnation point is $s + 0.25/s$ for $s < -0.5$ or $s > 0.5$, which implies that the slit makes the stagnation point shift by $0.25/s$ (or $0.25/s$). For $-0.5 < s < 0.5$, the saddle stagnation point is formed at the point (2s, 0) and the attachment point of the stagnation streamline is ($s + 0.25/s$, 0) beneath one of the plates. In this case, distance of the additional shift of the stagnation point due to the slit is s or $–s$ depending on the sign of s.

CONCLUSION

In this study, the effect of the centerline shift of the two-dimensional slow stagnation flow toward a slit is analytically investigated based on the Stokes approximation. A closed form solution for the flow such as the velocity, pressure, and stream function by using complex analytic functions. For the representative cases, the flow patterns which are constructed by numerically calculating the distribution of the stream function are presented and discussed in detail. The formation of viscous eddy and the separation of flow from the plate are also discussed by examining the velocity and the shear stress distributions.

The flow considered here can be realized when a viscous fluid flows normally past a finite flat plate with a crack. The present study is complementary to the study on the two-dimensional slow stagnation flow toward a slit in that a shift of the centerline of the approaching stagnation flow is allowed. In order to completely understand and utilize the characteristics of the flow past a flat plate with a crack, further theoretical, numerical, and experimental studies are needed. For example a stagnation flow in the presence of the pressure difference between two sides of plate is important since pressure usually differs between the sides. As it is common for a fluid to pass a flat plate obliquely, a study on the stagnation flow with oblique centerline is also needed.

ACKNOWLEDGMENTS

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