BINARY CONSTRUCTION OF QUANTUM CODES FROM SELF-DUAL ORIENTABLE EMBEDDINGS OF K_r, $K_{r,s}$ AND $K_{rs,s,s}$

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Abstract- In this paper we present four new classes of binary quantum codes based on self-dual orientable embeddings of complete, bipartite and tripartite graphs and by current graphs and rotation schemes. These classes of codes suggest a more general construction of codes defined on the closed orientable surfaces. We also show a table comparing the rate of these quantum codes when fixing the minimum distance to 3 and 4.

Keywords- Quantum Codes, Embedding, Self-Dual, Complete Graphs, Bipartite Graphs, Tripartite Graphs, Current Graphs, Rotation Schemes.

I. INTRODUCTION

Quantum error-correcting codes (QECs) play an important role in the theory of quantum information and computation. A main difficult to realize quantum computation is decoherence of quantum bits due to the interaction between the system and the surrounding environments. The QECs provide an efficient way to overcome decoherence. The first quantum code $[[9,1,3]]$ was discovered by Shor [1]. Calderbank et al. [2] introduced a systematic way for constructing the QECs from classical error-correcting code. The problem of constructing toric quantum codes has motivated considerable interest in the literature. This problem was generalized within the context of surface codes [8] and color codes [3]. The most popular toric code was proposed for the first time by Kitaev's [5]. This code defined on a square lattice of size $m \times m$ on the torus. The parameters of this class of codes are $[[n,k,d]]=[[2m^2,2,m]]$. In similar way, the authors in [7] introduced a construction of topological quantum codes in the projective plane RP^2. They showed that the original Shor's 9-qubit repetition code is one of these codes which can be constructed in a planar domain. Leslie proposed a new type of sparse CSS quantum error correcting codes based on the homology of hypermaps defined on an $m \times m$ square lattice [6]. The parameters of hypermap-homology codes are $[[(\frac{3}{2})m^2,2,m]]$. These codes are more efficient than Kitaev's toric codes. This seemed suggests good quantum that is constructed by using hypergraphs. But there are other surface codes with better parameters than the $[[2m^2,2,m]]$ toric code. There exist surface codes with parameters $[[m^2+1,2,m]]$, called homological quantum codes. These codes were introduced by Bombin and Martin-Delgado [8]. Authors in [9] presented a new class of toric quantum codes with parameters $[[m^2,2,m]]$, where $m=2(l+1), l \geq 1$. Sarvepalli [10] studied relation between surface codes and hypermap-homology quantum codes. He showed that a canonical hypermap code is identical to a surface code while a noncanonical hypermap code can be transformed to a surface code by CNOT gates alone. Li et al. [17] introduced a large number of good binary quantum codes of minimum distances five and six by Steane's Construction. In [18] good binary quantum stabilizer codes obtained via graphs of Abelian and non-Abelian groups schemes. In [19], Qian presented a new method of constructing quantum codes from cyclic codes over finite ring F_2+vF_2.

Our aim in this work is to present four new classes of binary quantum codes with parameters

(i)$[[2r(4r+1),2r(4r-3),3]]$, where $r \geq 1$,

(ii) $[[2s(4s-1),2(s-1)(4s-1),3]]$, where $s \geq 2$,

(iii) $[[rs,\frac{(r-2)(s-2)}{2},4]]$, where r and s are both divisible by 4,

(iv) $[[(2r+1)s^2,(rs-2)(s-1),3]]$, where $r,s \geq 2$,

respectively, based on results of Pengelley [13] and Archdeacon et al. [14] and Archdeacon [20] in self-dual orientable embeddings of the complete graphs K_{4r+1} and K_{4s}, ($r \geq 1$ and $s \geq 2$), complete bipartite and complete tripartite graphs by current graphs and rotation schemes [21]. Binary quantum codes are de_ned by pair (H_X, H_Z) of Z_2 -matrices with $H_X H_Z{}^T = 0$. These codes have parameters $[[n,k,d_{\min}]]$, where k logical qubits are encoded into n physical qubits with minimum distance $d_{\min}$. A minimum distance $d_{\min}$ code can

correct all errors up to $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$ qubits. The code rate for two classes of quantum codes of length $n = 2r(4r+1)$ and $n = 2s(4s-1)$ is determined by $\frac{k}{n} = \frac{2r(4r-3)}{2r(4r+1)}$ and $\frac{k}{n} = \frac{2(s-1)(4s-1)}{2s(4s-1)}$, and this rate approaches 1 as r and s tend to infinity.

The paper is organized as follows. The simplices definition, chain complexes and homology group are recalled in Section 2. In Section 3 we shall briefly present the current graphs and rotation schemes. In Section 4, we give a brief outline of self-dual orientable embeddings of the complete graph. In Section 5, quantum codes from graphs on surfaces are reviewed. In Section 6, four classes of binary quantum codes by using self-dual orientable embeddings of complete graphs K_{4r+1} and K_{4s}, complete bipartite and tripartite graphs are constructed. In Section 7, we present the quantum codes when fixing the minimum distance to 3 and 4 in Table 1. The paper is ended with a brief conclusion.

II. HOMOLOGICAL ALGEBRA

In this section, we review some fundamental notions of homology spaces. For more detailed information about homology spaces, refer to [4], [12].

Simplices. Let $m, n \in N, m \geq n$. Let moreover the set of points $\{v_0, v_1, ..., v_n\}$ of R^m be geometrically independent.

An n-simplex Δ is a subset of R^m given by

$$\Delta = \left\{ x \in R^m \middle| x = \sum_{i=0}^{n} t_i v_i; 0 \leq t_i \leq 1; \sum_{i=0}^{n} t_i = 1 \right\} \quad (2.1)$$

Chain complexes. Let K be a simplicial complex and p a dimension. An p-chain is a formal sum of p-simplices in K. The standard notation for this is $c = \sum_i n_i \sigma_i$, where $n_i \in Z$ and σ_i is a p-simplex in K. Let $C_p(K)$ be the set of all p-chains in K. The boundary homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ is defined as

$$\partial_k(\sigma) = \sum_{j=0}^{k} (-1)^j [v_0, v_1, ..., v_{j-1}, v_{j+1}, ..., v_k] \quad (2.2)$$

The chain complex is the sequence of chain groups connected by boundary homomorphisms,

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots \quad (2.3)$$

Cycles and boundaries. We are interested in two subgroups of $C_p(K)$, cycle and boundary groups. The $p-th$ cycle group is the kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$, and denoted $Z_p = Z_p(K)$. The $p-th$ boundary group is the image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$, and denoted as $B_p = B_p(K)$.

Definition 2.1 (Homology group, Betti number). The $p-th$ homology group H_p is the $p-th$ cycle group modulo the $p-th$ boundary group, $H_p = \frac{Z_p}{B_p}$. The $p-th$ Betti number is the rank (i.e. the number of generators) of this group, $\beta_p = rank\ (H_p)$. So the first homology group H_1 is given as

$$H_1 = \frac{Z_1}{B_1}. \quad (2.4)$$

From the algebraic topology, we can see that the group H_1 only depends, up to isomorphisms, on the topology of the surface [4]. In fact

$$H_1 \cong Z_2^{2g}, \quad (2.5)$$

where g is the genus of the surface, i.e. the number of "holes" or "handles". We then have

$$|H_1| = 2^{2g} \quad (2.6)$$

III. CURRENT GRAPHS

Current graphs were invented before voltage graphs. These graphs are used in proof of Map Color Theorem, determination of minimum genus of complete graphs. Current graphs are dual of voltage graphs which apply to embedded voltage graphs. Faces of current graph correspond to vertices in voltage graph, and vice versa. We refer the reader to [21] for more details.

3.1. Rotation schemes

Rotation schemes are important for applying the face-tracing. Let $G = (V, G)$ be a connected graph. Denote the vertex set of G by $V(G) = \{1, 2, ..., n\}$. For each $i \in V(G)$,

let $V(i) = \{k \in V(G) | \{i, k\} \in E(G)\}$.

Let $p_i : V(i) \to V(i)$ be an oriented cyclic order (or a cyclic permutation) on $V(i)$, of length

$n_i = |V(i)|$; p_i ia called a rotation scheme, or rotation system. By combining the two concepts of rotation schemes and 2-cell embeddings we have the following theorem:

Theorem 3.1. Every rotation scheme for a graph G induces a unique embedding of G into an orientable surface. Conversely, every embedding of a graph into orientable surface induces a unique rotation scheme for G.

Proof. The Proof of this theorem is found in [16].

Often it is customary to represent a graph G with rotation in the plane in such a way that a clockwise (or counterclockwise) reading of the edges incident with a vertex gives the rotation at that vertex. By convention, a solid vertex has its incident edges ordered clockwise; a hollow vertex, counterclockwise. For further information about rotation schemes, the reader is referred to Refs. [16, 24].

IV. SELF-DUAL ORIENTABLE EMBEDDINGS OF COMPLETE GRAPHS

Let S be a compact, connected, oriented surface (i.e. 2-manifold) with genus g. As shown by Pengelley [13], Euler's Formula excludes self-dual orientable embeddings of K_n unless $n \equiv 0$ or $1 \pmod 4$. Let T be a self-dual embedding of S and α_0, α_1, α_2 denote respectively the number of vertices, edges and faces T. If the embedding is in S, then

$$\alpha_2 = \alpha_0 = n, \qquad \alpha_1 = \frac{n(n-1)}{2} \tag{4.1}$$

and Euler's Formula follows that

$$g = \frac{2 - \alpha_0 + \alpha_1 - \alpha_2}{2} = \frac{(n-1)(n-4)}{4} \tag{4.2}$$

Hence such an embedding can exist only if $n \equiv 0$ or $1 \pmod 4$. Since the embedding consists of n faces, each face must be adjacent to every other along exactly one edge. Hence each face is an $(n-1)$-gon. Pengelley presented how to use current graphs and rotation schemes to describe an orientable embedding of K_n having n $(n-1)$-gons as faces, and then show such an embedding is self-dual [13]. Since we wish our faces to be $(n-1)$-gons, by the current graph construction principles is requiring that each vertex in the current graph be of valence $n-1$ [22].

In the case $n \equiv 1 \pmod 4$, we select a group Γ for which K_{4r+1} is a Cayley color graph; in this case, we can only pick $\Gamma = Z_{4r+1}$. Label the vertices K_{4r+1} with the elements of Γ. Denote the faces by the numbers $0,1,...,n-1$. Then choose a certain orientation for each face. Write down the cyclic order of the faces adjacent to face 0. This gives a certain permutation of $1,2,...,n-1$. Do the same for the other faces. This leads to a scheme. The scheme for K_{4r+1} is generated by the following 0 [13].

$$0.1,-2,-1,2,3,-4,-3,4,\ldots,2k-1,-2k,-(2k-1),$$
$$2k,..,2r-1,-2r,-(2r-1),2r \ (1 \le k \le r). \tag{4.3}$$

Explicit calculation gives the following cyclic sequence for the vertices of the face containing the directed edge from 0 to 1:

$$0,1,3,2,0,\ldots 0,2k-1,4k-1,2k,0,\ldots 0,2r-1,4r-1,2r$$
$$(1 \le k \le r). \tag{4.4}$$

In general, by explicit calculation from scheme, we obtain, for each $i \in Z_{4r+1}$, the cyclic sequence

$$i,1+i,3+i,2+i,i,\ldots,i,(2k-1)+i,(4k-1)+i,2k+i,$$
$$i,..,i,(2r-1)+i,(4r-1)+i,2r+i \ (1 \le k \le r). \tag{4.5}$$

for the vertices of the face containing the directed edge e_i, where e_i is an edge between vertices i and $i+1$.

In the case $n \equiv 0 \pmod 4$, Pengelly [13] has used the following group

$$G = \underbrace{Z_2 \times Z_2 \times \cdots \times Z_2}_{s} \times Z_t$$

where $n = 2^s t$, t is odd, and $s \ge 2$. There are precisely $2^s - 1$ elements of order two. Pengelly in Ref. [13] released $2^s(t-1)$ elements with distinct inverses, and his choose $a_1, a_2, a_3, \ldots, a_{2^{s-1}(t-1)}$ from this collection such that they and their inverses deplete the collection. We label the elements of order two $b_1, b_2, b_3, \ldots, b_{2^s-1}$. From the elementary group theory, we can see that $\sum_{l=1}^{2^s-1} b_l = 0$. The scheme for $K_{4s}(s \ge 2)$ is generated by the following 0 [13].

$$0.a_1,-a_2,-a_1,a_2,a_3,-a_4,-a_3,a_4,\ldots,a_{2k-1},-a_{2k},$$
$$-a_{2k-1},a_{2k},\ldots,a_{2^{s-1}(t-1)-1},-a_{2^{s-1}(t-1)},-a_{2^{s-1}(t-1)-1},a_{2^{s-1}(t-1)},$$
$$b_1,b_2,\ldots,b_{2^s-1}\ (1\le k\le 2^{s-2}(t-1)).\quad (4.6)$$

This generates by the additive rule the entire scheme for K_{4s}. Explicit calculation yield the following cyclic sequence for the vertices of the face containing the directed edge from 0 to a_1 [13]:

$$0,a_1,a_1+a_2,a_2,0,a_3,a_3+a_4,a_4,0,\ldots,0,a_{2k-1},$$
$$a_{2k-1}+a_{2k},a_{2k},0,\ldots,0,a_{2^{s-1}(t-1)-1}+a_{2^{s-1}(t-1)},a_{2^{s-1}(t-1)},$$
$$0,b_1,b_1+b_2,b_1+b_2+b_3,\ldots,\sum_{l=1}^{2^s-2}b_l$$
$$(1\le k\le 2^{s-2}(t-1)).\quad (4.7)$$

By calculation from scheme, we obtain, for each $g\in G$, the cyclic sequence for the vertices of the face containing the directed edge from g to $g+a_1$:

$$g,a_1+g,a_1+a_2+g,a_2+g,g,a_3+g,a_3+a_4+g,$$
$$a_4+g,g,\ldots,g,a_{2k-1}+g,a_{2k-1}+a_{2k}+g,g,\ldots,g,$$
$$a_{2^{s-1}(t-1)-1}+a_{2^{s-1}(t-1)}+g,a_{2^{s-1}(t-1)}+g,g,b_1+g,$$
$$b_1+b_2+g,b_1+b_2+b_3+g,\ldots,g+\sum_{l=1}^{2^s-2}b_l$$
$$(4.8)\ (1\le k\le 2^{s-2}(t-1)).$$

V. QUANTUM CODES FROM GRAPHS ON SURFACES

The idea of constructing CSS (Calderbank-Shor-Steane) codes from graphs embedded on surfaces has been discussed in a number of papers. See for detailed descriptions e.g. [11]. Let X be a compact, connected, oriented surface (i.e. 2-manifold) with genus g. A tiling of X is defined to be a cellular embedding of an undirected (simple) graph $G=(V,E)$ in a surface. This embedding defines a set of faces F. Each face is described by the set of edges on its boundary. This tiling of surface is denoted $M=(V,E,F)$. The dual graph G is the graph $G^*=(V^*,E^*)$ such that:

(i) One vertex of G^* inside each face of G,

(ii) For each edge e of G there is an edge e^* of G^* between the two vertices of G^* corresponding to the two faces of G adjacent to e.

It can easily see that, there is a bijection between the edges of G and the edges of G^*.

The surface code associated with a tiling $M=(V,E,F)$ is the CSS code defined by the matrices H_X and H_Z such that $H_X\in \mathrm{M}_{|V|,|E|}(Z_2)$ is the vertex-edge incidence matrix of the tiling and $H_Z\in \mathrm{M}_{|F|,|E|}(Z_2)$ is the face-edge incidence matrix of the tiling. Therefore, from (X,G) is constructed a CSS code with parameters $[[n,k,d]]$. where n is the number of edges of G, $k=2g$ (by (2.6)) and d is the shortest non-boundary cycle in G or G^*. In this work, the minimum distance of quantum codes by a parity check matrix H (or generator matrix) is obtained. For a detailed information to compute the minimum distance, we refer the reader to [15].

VI. CODE CONSTRUCTION

In this section, we construct four classes of binary quantum codes based on self-dual orientable embeddings of complete, bipartite and tripartite graphs and by current graphs and rotation schemes.

6.1. Construction I

In this section we construct binary quantum codes from self-dual orientable embeddings of complete graph K_{4r+1} of length $n=2r(4r+1), r\ge 1$.

Let $G=K_m$ be an embedding in an orientable surface S. We know that $|E(G)|=\dfrac{m(m-1)}{2}$.

Also, from (4.2) with a self-dual embedding of complete graph on an orientable surface of genus g, we know that if $m=4r+1\equiv 1 \pmod 4$, then

$$g=\frac{(m-1)(m-4)}{4}=r(4r-3).$$

Therefore, $|E(G)|=2r(4r+1)$. By finding the vertex-edge incidence matrix H_X using the relation (4.3) and rotation schemes, and the face-edge incidence matrix H_Z by using (4.5), one can easily see that $H_X H_Z{}^T=0$ and $d_{\min}=3$.

Thus the code parameters are given by: the code minimum distance is $d_{\min}=3$; the code length is $|E(G)|=2r(4r+1)$ and $k=2g=2r(4r-3)$.

Consequently, the class of codes with parameters $[[2r(4r+1),2r(4r-3),3]]$ is obtained.

Example 6.1.1. Let $m = 5 = 4 \times 1 + 1 \equiv 1 (\text{mod } 4)$. Then $n = |E(G)| = 10$ and $k = 2g = 2.$ In Figure 1 we give a self-dual embedding of K_5 into the torus. The torus is recovered from the rectangle by identifying its left and right sides and simultaneously identifying its top and bottom sides.

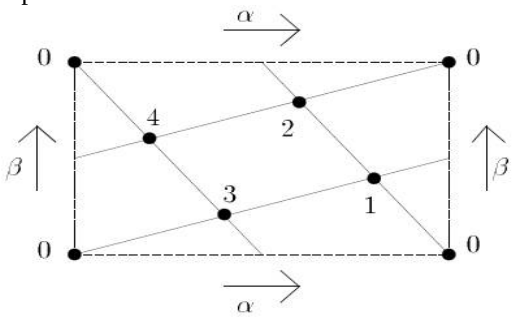


Figure 1: A rotation embedding K_5 in the torus

In this figure we have:

0.	2	4	3	1
1.	3	0	4	2
2.	4	1	0	3
3.	0	2	1	4
4.	1	3	2	0

By rotation schemes, we get the following vertex-edge and face-edge matrices respectively:

$$H_X = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$H_Z = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

One can easily see that $H_X H_Z{}^T = 0$ and $d_{\min} = 3.$ Therefore, the code with parameters [[10,2,3]] is obtained.

Example 6.1.2 Let $m = 9 = 4 \times 2 + 1 \equiv 1$ (mod 4). Then $n = |E(G)| = 36$ and $k = 2g = 20$. For determining H_X and H_Z by using (4.3) and rotation schemes, we have:

0.	1	7	8	2	3	5	6	4
1.	2	8	0	3	4	6	7	5
2.	3	0	1	4	5	7	8	6
3.	4	1	2	5	6	8	0	7
4.	5	2	3	6	7	0	1	8
5.	6	3	4	7	8	1	2	0
6.	7	4	5	8	0	2	3	1
7.	8	5	6	0	1	3	4	2
8.	0	6	7	1	2	4	5	3

(6.1)

Also, by using (4.4) and (4.5), we obtain the following cyclic sequence for the vertices of the face containing the directed edge for each $i \in Z_9$

(0)	0	1	3	2	0	3	7	4
(1)	1	2	4	3	1	4	8	5
(2)	2	3	5	4	2	5	0	6
(3)	3	4	6	5	3	6	1	7
(4)	4	5	7	6	4	7	2	8
(5)	5	6	8	7	5	8	3	0
(6)	6	7	0	8	6	0	4	1
(7)	7	8	1	0	7	1	5	2
(8)	8	0	2	1	8	2	6	3

(6.2)

By using (6.1) and (6.2) we obtain the following vertex-edge and face-edge matrices respectively:

$H_X = \langle A_1 | A_2 \rangle$

where

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$H_Z = \langle B_1 | B_2 \rangle$

where

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&1&0&0&1&0&0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&1&1&1&1&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&1&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

One can easily see that $H_X H_Z{}^T = 0$ and $d_{\min} = 3.$ Therefore, the code with parameters $[[36,20,3]]$ is obtained.

6.2. Construction II

In this section we construct binary quantum codes from self-dual orientable embeddings of complete graph K_{4s} of length $n = 2s(4s-1), s \geq 2.$

Let $G = (V, E)$ be a self-dual graph on $4s$ vertices. From (4.1), we know that $|E(G)| = 2s(4s-1).$ Also, from (4.2) we know that $g = (s-1)(4s-1).$ Therefore, the code length is $n = |E(G)| = 2s(4s-1)$ and $k = 2g = 2(s-1)(4s-1).$ With finding the matrices H_X and H_Z using the relations (4.6), (4.7) and (4.8), one can see that $H_X H_Z{}^T = 0$ and $d_{\min} = 3.$ Consequently, the class of codes with parameters $[[2s(4s-1),2(s-1)(4s-1),3]]$ is constructed.

6.3. Construction III

In this section we construct binary quantum codes from self-dual orientable embeddings of complete bipartite graph $K_{r,s}$ of length $n = rs.$

Let the complete bipartite graph $G = K_{r,s}$ be an embedding in S_g (the orientable 2-manifold of genus g). The graph $K_{r,s}$ has $r+s$ vertices divided into two subsets, one of size r and the other of size s. The number of edges in a complete bipartite graph $K_{r,s}$ is $|E| = rs.$ From Ref. [23] and the following theorem, the genus of the complete bipartite graph $K_{r,s}$ is given as

$$g = \frac{(r-2)(s-2)}{4} \qquad (6.3)$$

where r and s are both divisible by $4.$

Theorem 6.3.1. $K_{r,s}$ has both an orientable and a nonorientable self-dual embedding for all even integers r and s exceeding 2, except that there is no orientable self-dual embedding of $K_{6,6}.$

Proof. The Proof of this theorem is found in [14].

In the complete bipartite graph $G = K_{r,s}$, where r and s are both divisible by 4, we know that $|E(G)| = rs.$ Since in this self-dual embedding on an orientable surface the code minimum distance is four. Thus, the code parameters are given by: the code minimum distance $d_{\min} = 4$; the code length is $n = |E(G)| = rs$ and $k = 2g = \frac{(r-2)(s-2)}{2}.$ Consequently, the new class of codes with parameters $[[rs, \frac{(r-2)(s-2)}{2}, 4]]$ is obtained.

Example 6.3.1 In Figure 2 we give a self-dual embedding of K4;4 into the torus. In this figure the top of the rectangle is identified with the bottom and the left with the right to recover the torus.

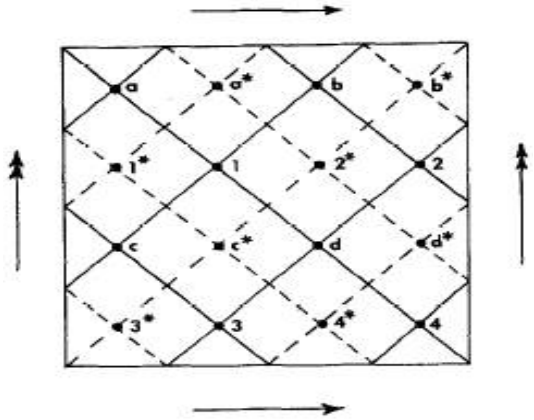


Figure 2: A self-dual embedding of $K_{4,4}$ in the torus

In this figure the primal graph is shown by solid lines, while the dual graph is drawn with dotted lines. The vertex bipartitions are $\{a,b,c,d\}$ vs. $\{1,2,3,4\}.$ Since in this figure the shortest non-boundary cycle in the primal graph or the dual graph is 4. So, $d_{\min} = 4.$

Also, we have $n = 4 \times 4 = 16$, the number of edges and $k = 2g = 2 \times \frac{(4-2)(4-2)}{4}$ (by (6.3)). Hence, a $[[16,2,4]]$ code is obtained.

6.4. Construction IV

In this section we construct binary quantum codes from self-dual orientable embeddings of complete bipartite graph $K_{rs,s,s}$ of length $n = (2r+1)s^2.$

Let $G = K_{rs,s,s}$ be a complete tripartite embedded on a closed surface. The graph $K_{rs,s,s}$ has $rs + s + s$ vertices. The number of edges in a complete tripartite graph $K_{rs,s,s}$ is $|E| = rss + rss + s^2 = (2r+1)s^2$. From Ref. [26], the genus of the complete tripartite graph $K_{rs,s,s}$ is given as

$$g = \frac{(rs-2)(s-1)}{2} \qquad (6.4)$$

Theorem 6.4.1. The complete multipartite graph $K_{m(n)}, m \geq 2$, has both an orientable and a nonorientable self-dual embeddings except in the following cases:

(1) When the surface is orientable: n is odd and $m \equiv 2$ or $3 \pmod 4$, or when the graph is $K_{6,6}$, or possibly when the graph is $K_{4(2)}$ or $K_{6(2)}$.

(2) When the surface is nonorientable: the graph is K_n, where $n \leq 5$, or when the graph is $K_{6(3)}$.

Proof. The Proof of this theorem is found in [20].

In the complete tripartite graph $K_{rs,s,s}$, we know that $|E| = (2r+1)s^2$. Since in this self-dual embedding (except the Case 1 of the above theorem) on an orientable surface the code minimum distance is three. Thus, the code parameters are given by: the code minimum distance $d_{\min} = 3$; the code length is $n = |E(G)| = (2r+1)s^2$ and $k = 2g = (rs-2)(s-1)$. Consequently, the new class of codes with parameters $[[(2r+1)s^2, (rs-2)(s-1), 3]]$ is constructed.

VI. TABLE OF CODES QUANTUM

In this section, we present a table of binary quantum codes when fixing the minimum distance to 3 and 4.

Table1: Quantum codes from self-dual orientable embeddings of K_r, $K_{r,s}$ and $K_{rs,s,s}$

n	k	$d_{\min}$	rate
$2r(4r+1)$	$2r(4r-3)$	3	approaches to 1
$2s(4s-1)$	$2(s-1)(4s-1)$	3	approaches to 1
rs	$\frac{(r-2)(s-2)}{2}$	4	approaches to $\frac{1}{2}$
$(2r+1)s^2$	$(rs-2)(s-1)$	3	approaches to $\frac{1}{2}$

CONCLUSION

We considered presentation of four new classes of binary quantum codes based on self-dual orientable embeddings of the complete graphs K_{4r+1} and K_{4s}, ($r \geq 1$ and $s \geq 2$), complete bipartite graph $K_{r,s}$, and complete tripartite graph $K_{rs,s,s}$ by using current graphs and rotation schemes. These codes are superior to quantum codes presented in other references. We point out the classes $[[2r(4r+1), 2r(4r-3), 3]]$ and $[[2s(4s-1), 2(s-1)(4s-1), 3]]$ of quantum codes achieve the best ratio $\frac{k}{n}$. For new classes $[[rs, \frac{(r-2)(s-2)}{2}, 4]]$ and $[[(2r+1)s^2, (rs-2)(s-1), 3]]$ of codes, the code rate $\frac{k}{n}$ approaches to $\frac{1}{2}$.

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